Introduction to Fuzzy Systems

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LŌŌ KA SA WITCH

Inference

* Multivalued Logic

⇒ Łukasiewicz Logic

Classic Boolean

a	b	⇒ ∧	∨	⇒
0	0	0	0	1
0	1	0	1	0
1	0	0	1	0
1	1	1	1	1

Łukasiewicz Logic, L_3 ('The Logic of Half Truths')

a	b			
0	0	0	0	1
0	½	0	½	½
0	1	0	1	0
½	0	0	½	½
½	½	½	½	1
½	1	½	1	½
1	0	0	1	0
1	½	½	1	½
1	1	1	1	1
a	b	∧	∨	⇒

a	$\bar{a}$
0	1
½	½
1	0

Notes:

① L_n is Łukasiewicz Logic, order n

CONTINUM ∞ (George Cantor)

$L_{\aleph_1}$ is fuzzy logic (not $L_{\aleph_0}$)

② Meets Boundary Condition

③ ∨ is max, ∧ is min

$\bar{a} = 1 - a$

General Fuzzy Complement

* Sugeno Class

$$C_\lambda(a) = \frac{1-a}{1+\lambda a} \quad , \quad -1 < \lambda < \infty$$

$$C_0(a) = 1-a$$

Involutive?

$$C_\lambda[C_\lambda(a)] = \frac{1-C_\lambda(a)}{1+\lambda C_\lambda(a)}$$

$$= \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} = \frac{(1+\lambda a)-(1-a)}{(1+\lambda a)+\lambda(1-a)}$$

$$= \frac{\lambda a + a}{1+\cancel{\lambda a}+\lambda-\cancel{\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$$

p. 42

General Fuzzy Complements

Yager Class

$$C_W(a) = (1 - a^W)^{\frac{1}{W}} \quad ; \quad 0 < W < \infty$$

$$C_1(a) = 1 - a$$

$$C_W(0) = 1$$

$$C_W(1) = 0$$

Involutive?

$$C_W[C_W(a)] = (1 - C_W(a)^W)^{\frac{1}{W}}$$

$$= a$$

P 42

General Fuzzy Unions

Yager class

$$u_w(a,b) = \min\left[1, (a^w + b^w)^{1/w}\right] \quad ; 0<w<\infty$$

Satisfies all axioms <u>but</u> is not idempotent

Note

$$u_\infty(a,b) = \max(a,b) \Leftarrow \text{Zadeh Union}$$

Proof: Let

$$\xi = (a^w + b^w)^{1/w}$$

Then

$$\lim_{w\to\infty} \ln\xi = \frac{\ln(a^w + b^w)}{w}$$

$$= \lim_{w\to\infty} \frac{d}{dw} \ln(a^w + b^w)$$

$$= \lim_{w\to\infty} \frac{\frac{d}{dw}\left(e^{w\ln a} + e^{w\ln b}\right)}{a^w + b^w}$$

$$= \lim_{w\to\infty} \frac{(\ln a)a^w + (\ln b)b^w}{a^w + b^w}$$

$$= \lim_{w\to\infty} \frac{\ln a}{1 + (b/a)^w} + \frac{\ln b}{1 + (\frac{a}{b})^w}$$

$$= \begin{cases} \ln a & ; \quad b<a \\ \ln b & ; \quad b>a \end{cases}$$

$$= \max[\ln a, \ln b]$$

$$\Rightarrow \xi = \max[a,b]$$

Sum-Product Inferencing

$u(a,b) = \min[1, a+b] \leftarrow$ Yager with $w = 1$

General Fuzzy Intersections

Zadeh: $i(a,b) = \min(a,b); 0<w<\infty$

Yager

Not idempotent

$$i_w(a,b) = 1 - \min\left[1, \{(1-a)^w + (1-b)^w\}^{\frac{1}{w}}\right]$$

Note: $\hat{i}_\infty(a,b) = \min[a,b]$

Associative:

$$i_w[i_w(a,b), c] = 1 - \min\left[1, \{1 - i_w(a,b)\}^w + (1-c)^w\}^{\frac{1}{w}}\right]$$

$$\overset{?}{=} i_w[a, i_w(b,c)] = 1 - \min\left[1, \{1-a\}^w + (1 - i_w(b,c)^w\}^{\frac{1}{w}}\right]$$

Sum-Product

$i(a,b) = ab$

Boundary ✓
Commutative ✓
Monotonic ✓
Associative ✓
Continuous ✓
Idempotent X

$u(a,b) \geq \max(a,b)$

Proof: $u(a, u(0,0)) = u(u(a,0),0)$ ASSOC

$$\Rightarrow u(a,0) = u(u(a,0),0)$$

Assume $u(a,0) = \alpha \neq a$

$$\alpha = u(u(a,0),0) = u(\alpha,0)$$

$\Rightarrow$ CONTRADICTS: $u(\alpha,0) \neq \alpha$

Only solution is thus

$$u(a,0) = a$$

By Monotonicity

$$u(a,b) \geq u(a,0) = a$$

Commutivity

$$u(b,a) = u(a,b) \geq u(b,0) = b$$

Thus: $u(a,b) \geq \max(a,b)$

Generalized Means (meets all axioms)

$$h_\alpha(a_1, a_2, \ldots, a_n) = \left[\frac{1}{n}\left(a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha\right)\right]^{\frac{1}{\alpha}}$$

$; -\infty < \alpha < \infty$

Special Cases:

$h_{-\infty}(a_1, \ldots, a_n) = \min[a_1, \ldots, a_n]$

$h_\infty(a_1, \ldots, a_n) = \max[a_1, \ldots a_n]$

$h_0(a_1, \ldots, a_n) = (a_1 a_2 \ldots a_n)^{1/n} \Leftarrow$ GEOMETRIC MEAN

Proof:

$$\ln h_\alpha(a_1, \ldots, a_n) = \frac{1}{\alpha} \ln \frac{1}{n} \sum_{m=1}^n a_m^\alpha$$

$$= \frac{1}{\alpha}\left[\ln \sum_{m=1}^n a_m^\alpha - \ln n\right]$$

l'Hospital:

$$\lim_{\alpha \to 0} \ln h_\alpha(a_1, \ldots, a_n) = \lim_{\alpha \to 0} \frac{d}{d\alpha} \ln \sum_{m=1}^n a_m^\alpha$$

$$= \lim_{\alpha \to 0} \frac{\sum_{m=1}^n (\ln a_m) a_m^\alpha}{\sum_{m=1}^n a_m^\alpha}$$

$$= \sum_{m=1}^n \ln a_m / n$$

$$= \ln\left(\prod_{m=1}^n a_m\right) / n$$

$$\Rightarrow h_0(a_1, \ldots, a_n) = \left(\prod_{m=1}^n a_m\right)^{\frac{1}{n}}$$

$h_1(a_1 \ldots a_n) = \frac{1}{n} \sum_{m=1}^n a_m \Leftarrow$ ARITHMETIC MEAN

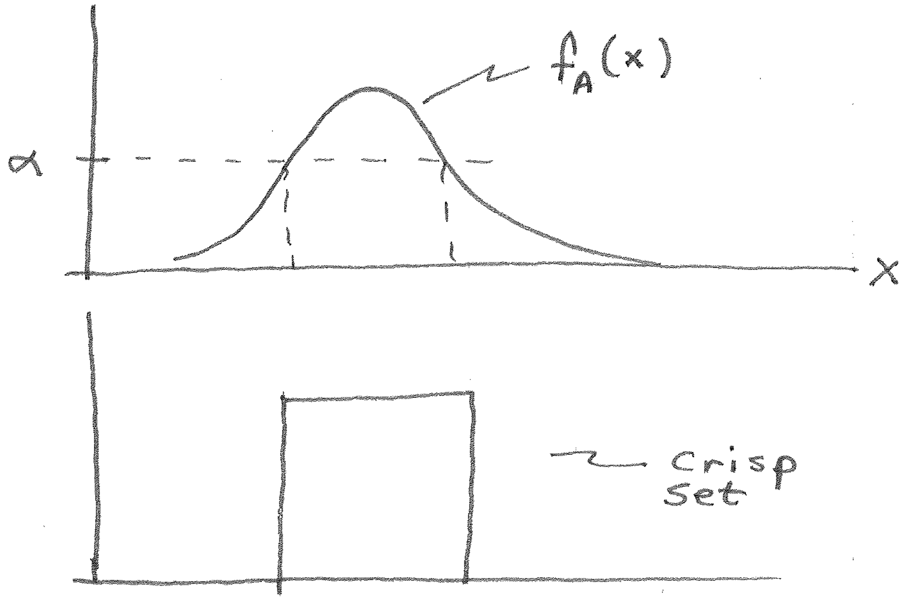
$h_{-1}(a_1, \ldots a_n) = \dfrac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}} \Leftarrow$ HARMONIC MEAN

Note:

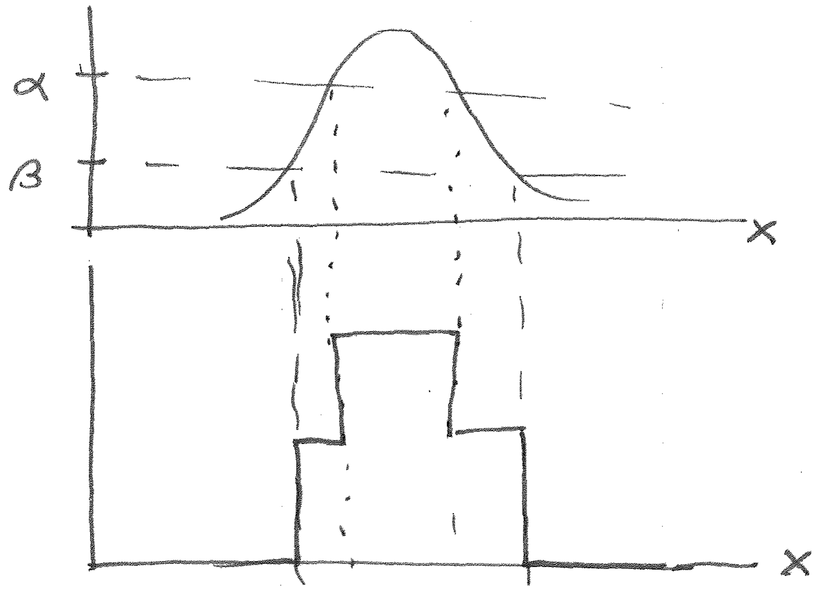
$$\min \le h_\alpha \le \max$$

α - cuts

Used to make fuzzy sets crisp



α, β cuts ⇒ Three-valued logic



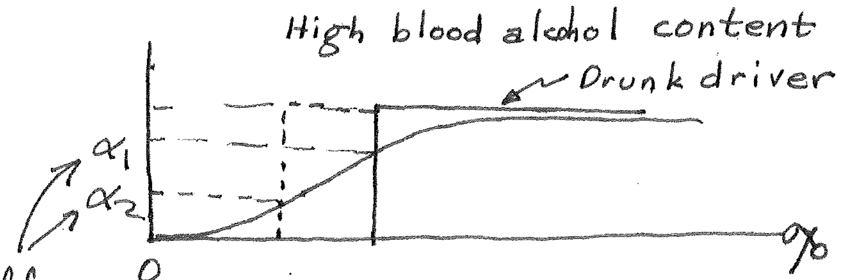
Extentions to multi-valued logic obvious.

Fuzzy = ∞ value logic ?

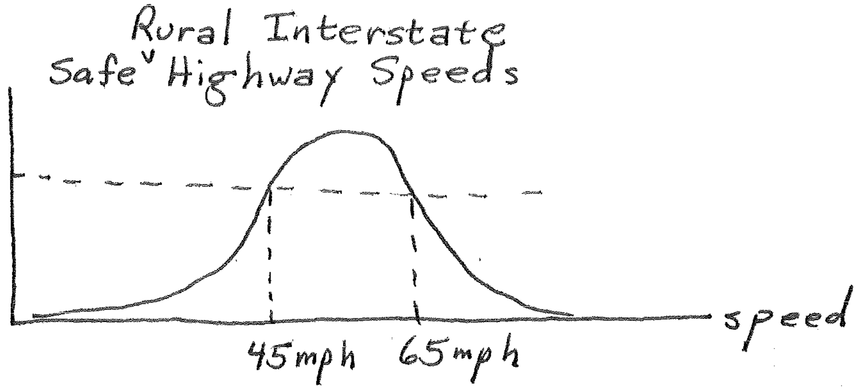
Note:

Situations are fuzzy

⇒ Law is crisp α cut:



Different States have different α cuts.



etc

(c) Convex Combination

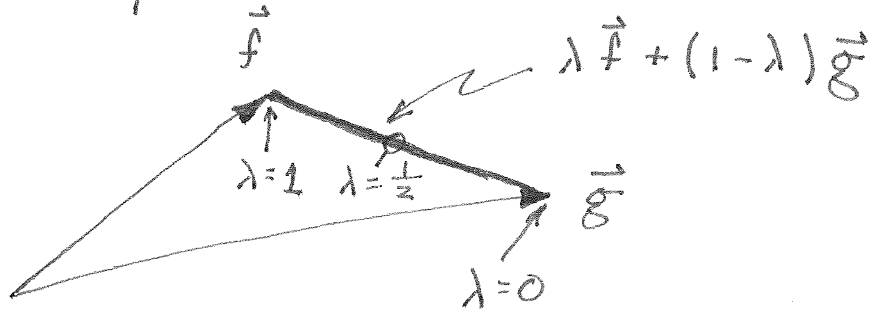
* Background:

Two vectors: $\vec{f}$ and $\vec{g}$

Convex Combination:

$$\lambda \vec{f} + (1-\lambda)\vec{g} \quad ; \quad 0 \le \lambda \le 1$$

Interpretation:



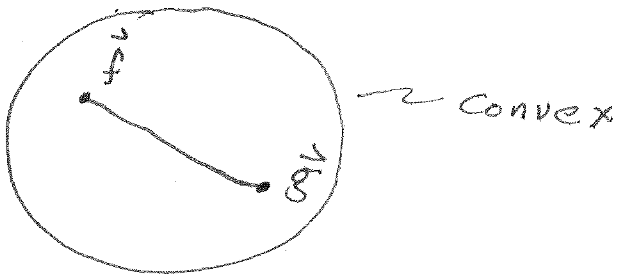
smallest set containing both $\vec{f}$ & $\vec{g}$

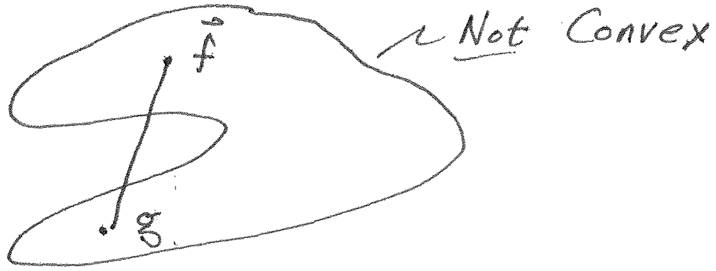
Definition of (crisp) convex set.

$\mathbb{C} = \{\vec{x}\}$ is convex if

$\forall\, \vec{f} \in \mathbb{C}$ and $\vec{g} \in \mathbb{C}$

$\Rightarrow \lambda \vec{f} + (1-\lambda)\vec{g} \in \mathbb{C} \ \forall\ 0 \le \lambda \le 1$





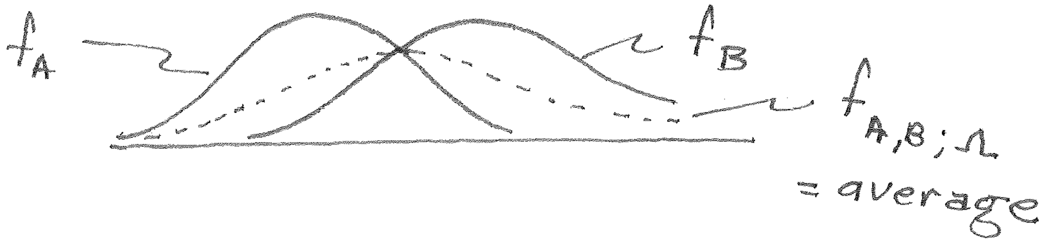
*Fuzzy Convex Combination of f_A and f_B"

$$(A,B;\Lambda) = \Lambda A + \bar{\Lambda} B$$

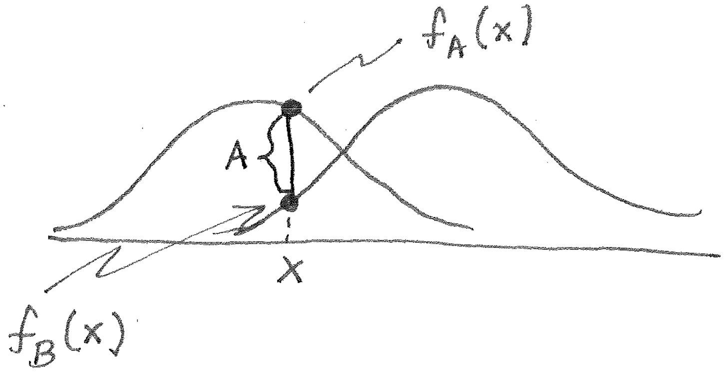
where Λ is a fuzzy membership function

$$f_{(A,B;\Lambda)}(x) = f_\Lambda(x) f_A(x) + (1 - f_\Lambda(x)) f_B(x)$$

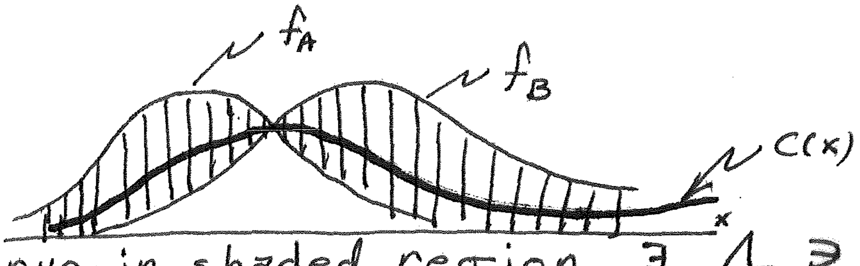
Example: $f_\Lambda = \frac{1}{2} \ \forall x$



The set of points $\forall \Lambda$:

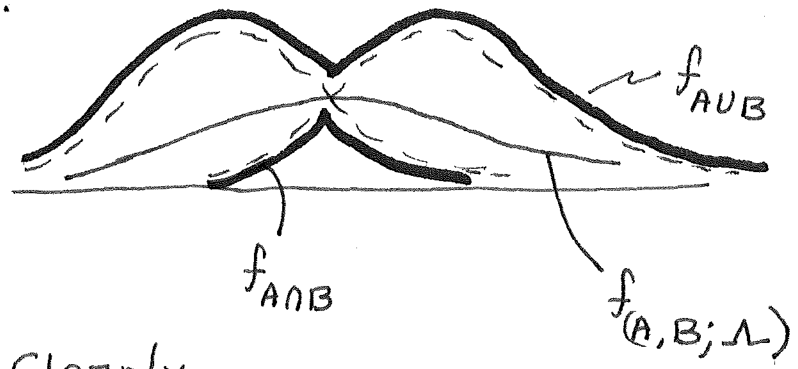


Since $0 \leq \Lambda \leq 1$, all points in interval A will be filled using all possible Λ's.



$\forall$ curve $C(x)$ in shaded region, $\exists \Lambda \ni$

$$f_{(A,B;\Lambda)}(x) = C(x)$$



Clearly

$$A \cap B \subset (A, B; \Lambda) \subset A \cup B \quad \forall \Lambda$$

Follows from:

$$\min[f_A(x), f_B(x)]$$
$$\leq \lambda f_A + (1-\lambda) f_B(x) \leq \max[f_A(x), f_B(x)]$$
$$\ni 0 \leq \lambda \leq 1 \quad \forall x$$

Note:

If

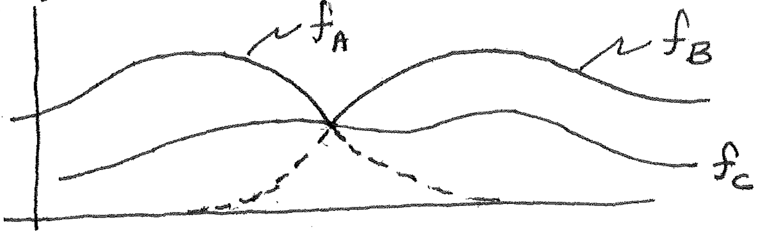
$$A \cap B \subset C \subset A \cup B$$

then $\exists \; \Lambda \; \ni$

$$C = (A, B; \Lambda)$$

Proof:

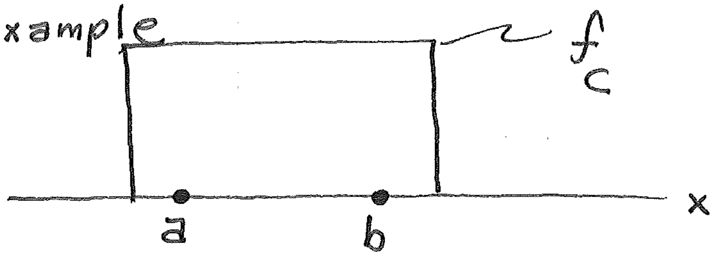
[redacted]



(e) Convexity

* For Crisp Sets:

- 1-D Example



$a \in C, \ b \in C \Rightarrow \alpha a + (1-\alpha) b \in C ; 0 \leq \alpha \leq 1$

- 2-D Example

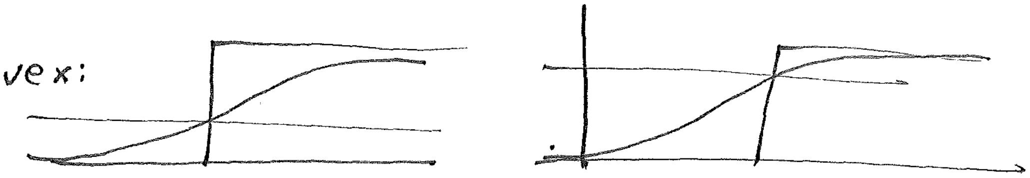
*For Fuzzy sets:

Fuzzy set A is convex if all of its α cuts are convex:

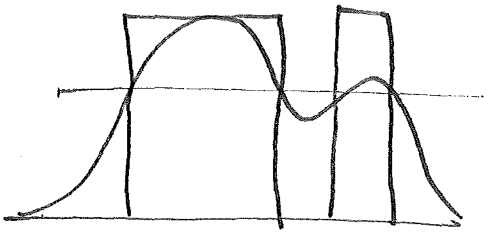
- 1-D

convex:

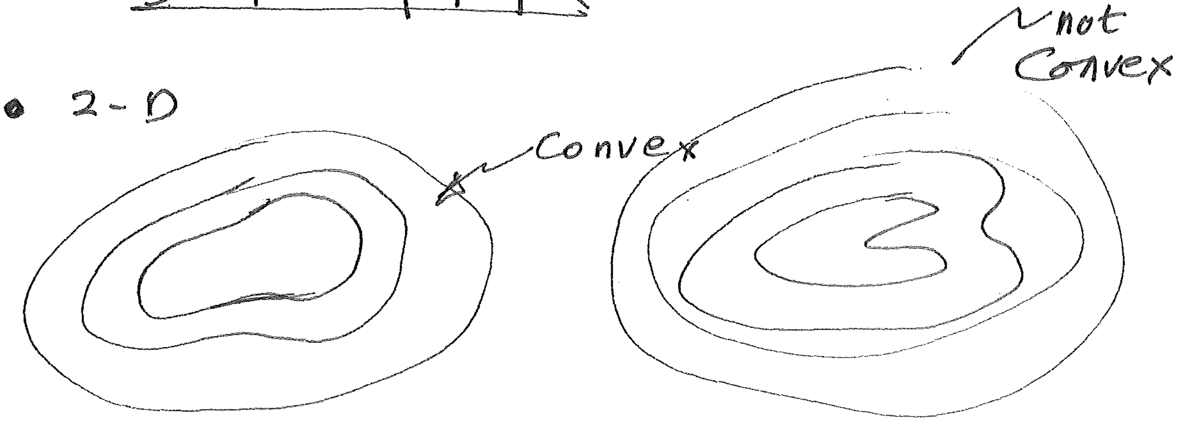
convex:



not covex



- 2-D

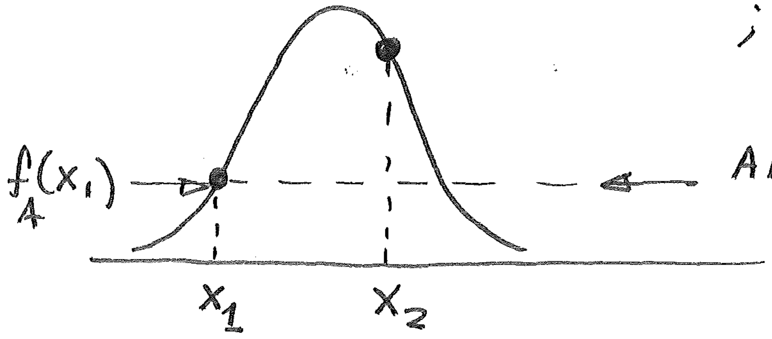


Alternate Definition:

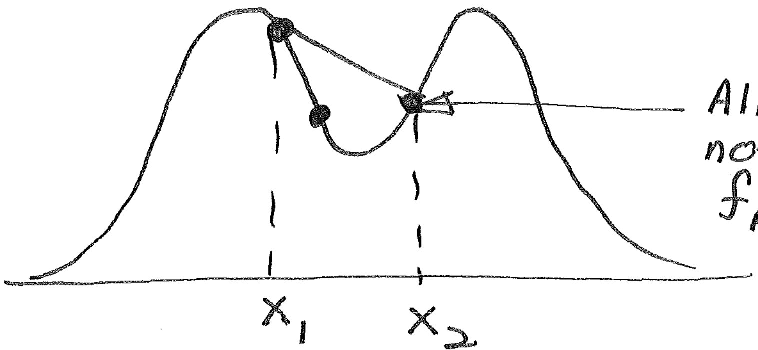
A is convex iff

$$f_A\left[\lambda x_1 + (1-\lambda) x_2\right] \geq \min\left[f_A(x_1), f_A(x_2)\right]$$

$; 0 \leq \lambda \leq 1$



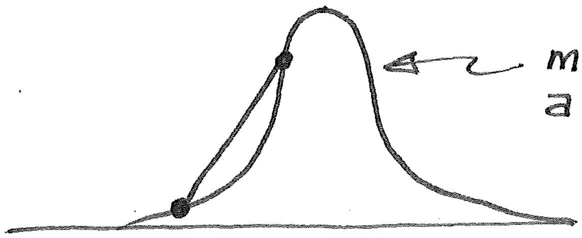
All points $x_1 \leq x \leq x_2$ lie above minimum, $f_A(x_1)$



All points do not lie above $f_A(x_2)$
$\Rightarrow$ not convex

Note: $f_A(x)$ need not be convex function. f_A is convex function, then

$$\lambda f_A(x_1) + (1-\lambda) f_A(x_2) \leq \cancel{\max} f_A(x)$$



membership for a convex set
$\neq$ convex function.

Q: Are all convex functions ('twixt 0 & 1) membership functions for fuzzy convex sets?

15

Example:

$$f_A(\vec{x}) = g(\overset{\geq 0}{\|\vec{x}\|}) \quad ; \quad \|\vec{x}\|^2 = \sum_{n=1}^{N} x_n^2$$

MONO ⇓

$$g(z) \Rightarrow g(z+\Delta) \cancel{\geq} \leq g(z) \quad \forall \; z>0, \Delta>0$$

$$\text{Let } \|\vec{x}_1\| > \|\vec{x}_2\| \Rightarrow f_A(\vec{x}_1) \leq f(\vec{x}_2)$$

Submit: A is convex

$$f_A(\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2) \geq f(\vec{x}_1) \, ? \quad ; 0 \leq \lambda \leq 1$$

Consider

$$\|\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2\| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{inequality}}}{\leq} \|\lambda \vec{x}_1\| + \|(1-\lambda)\vec{x}_2\|$$

$$= \lambda \|\vec{x}_1\| + (1-\lambda)\|\vec{x}_2\|$$

$$\leq \lambda \|\vec{x}_1\| + (1-\lambda)\|\vec{x}_1\| \quad (\text{Since } \|x_1\| > \|x_2\|)$$

$$= \|\vec{x}_1\|$$

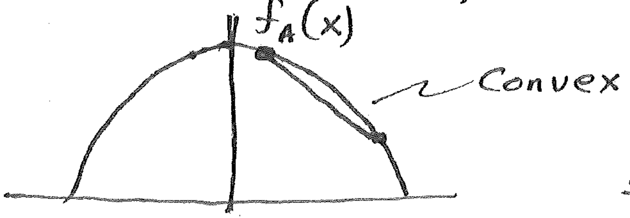
Thus

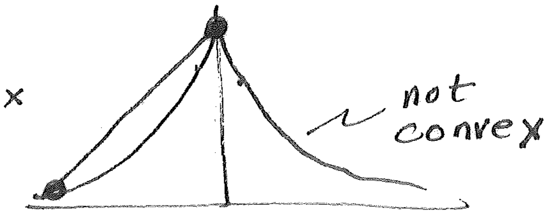
$$f_A[\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2] = g(\|\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2\|)$$

$$\geq g(\|\vec{x}_1\|) = f_A(\vec{x}_1)$$

∴ A is Convex

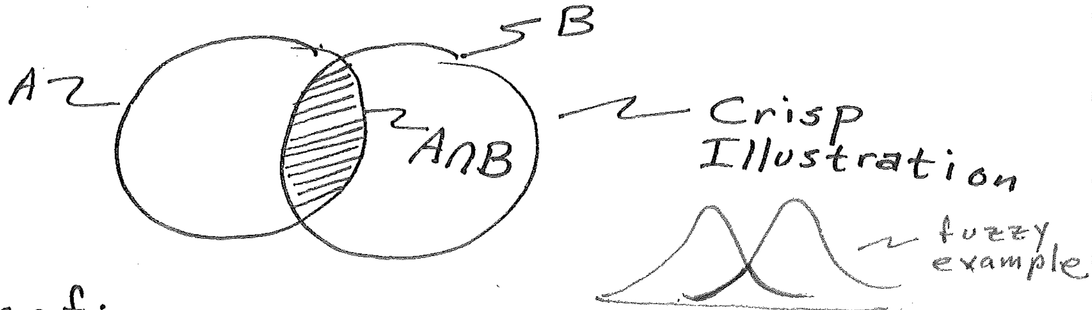
$f_A(\vec{x})$ may or may not be a convex function





* PROPERTIES

(i) A is convex fuzzy set,
B is convex fuzzy set
$\Rightarrow C = A \cap B$ is convex fuzzy set



Proof:

$$f_C[\lambda x_1 + (1-\lambda)x_2]$$

$$= \min\left[f_A(\lambda x_1 + (1-\lambda)x_2), f_B(\lambda x_1 + (1-\lambda)x_2)\right]$$

$$\geq \min\left[\min\left(f_A(x_1), f_A(x_2)\right), \min\left(f_B(x_1), f_B(x_2)\right)\right]$$

$$= \min\left[f_A(x_1), f_A(x_2), f_B(x_1) f_B(x_2)\right]$$

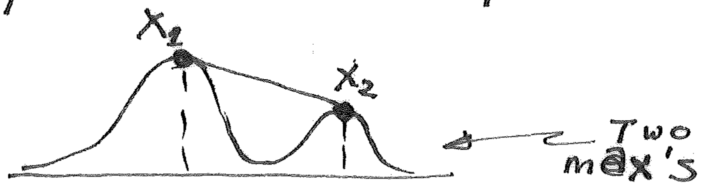
$$= \min\left[\min\left(f_A(x_1), f_B(x_1)\right), \min\left(f_A(x_2), f_B(x_2)\right)\right]$$

$$= \min\left[f_C(x_1), f_C(x_2)\right]$$

QED

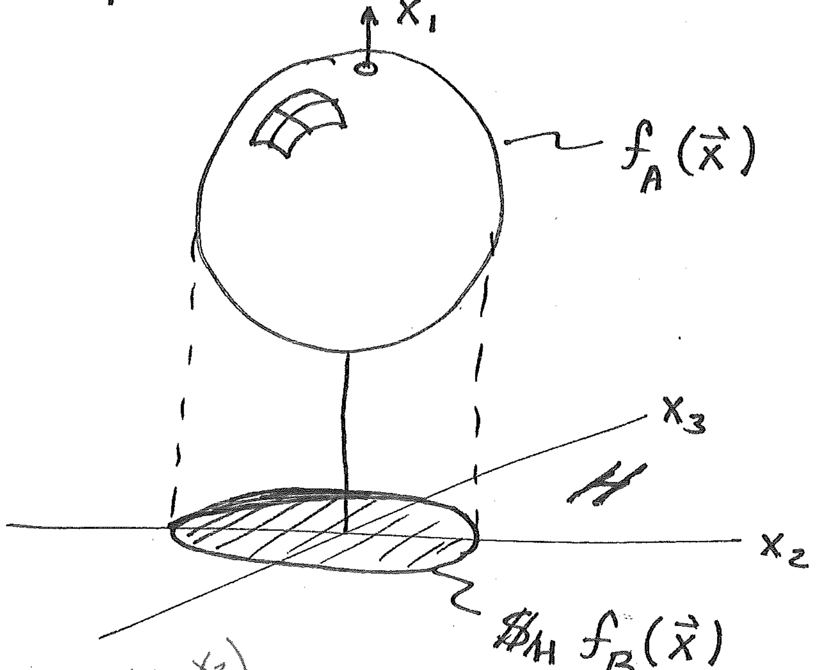
(ii) If A is a convex fuzzy set, it (f_A) has at most one maximum (supremum)

Proof by counterexample:



(g) Shadow

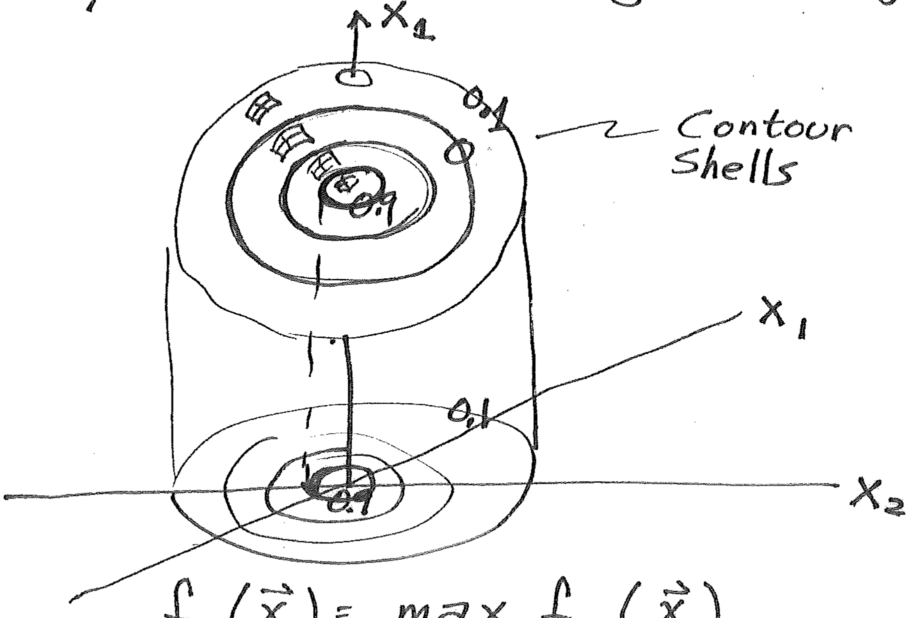
'Crisp Shadow'



$$B = S_H(A) = \text{SHADOW ON Plane } H$$

Note: $$f_B(\overset{(x_2,x_3)}{\vec{x}}) = \max_{x_1} f_A(\underbrace{\vec{x}}_{(x_1,x_2,x_3)})$$

'Fuzzy' Shadow (Analagous to 'Projection')



$$f_B(\vec{x}) = \max_{x_1} f_A(\vec{x})$$

Shadow of a

Convex Set; A

$f_A(\vec{x}) = f_A(x_1, x_2, \ldots, x_N)$

Shadow of A on plane $H = \{\vec{x} \mid x_1 = 0\}$ is B

$f_B(x_2, x_3, \ldots, x_N\} = \sup_{x_1} f_A(\vec{x})$

Zadeh's notation: $B = S_H(A)$

Note: If A is convex, so is B.

Theorem:

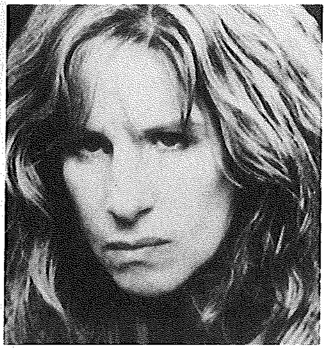
If $S_H(A) = S_H(B) \quad \forall$ H's,

then $A = B$

(Tomographic Shadow!)

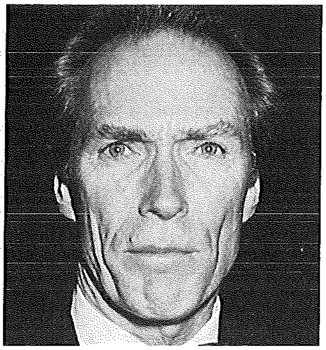
Conventional Projection

$$f_B(x_2, \ldots, x_N) = \int_{x_1} f_A(\vec{x})\, dx_1 \quad \left(\begin{matrix}\text{Abel} \\ \text{Transform}\end{matrix}\right)$$

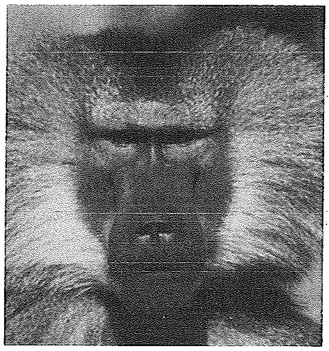


Barbra Streisand

The Beast



Clint Eastwood



a primate

$$M_R = \begin{array}{c} \begin{bmatrix} 1 & .8 & .2 \\ .8 & 1 & .2 \\ .2 & .2 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \nwarrow \\ \text{Barbra} & \text{Beast} & \text{Primate} \end{array} \end{array} \Leftarrow \text{fuzzy similarity relation}$$

Note:
1. Reflexive
2. Symm.
Transitive.

$@\ \alpha = 0.2 \Rightarrow$ everybody resembles everybody

$$@\ \alpha = 0.8 \Rightarrow M_{R_{0.8}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X/R_{0.8} = \{\{\text{Barbra}, \text{Beast}\}, \{\text{Primate}\}\}$$

(i) → Composition of two Fuzzy Relations[20]

fancy for max (elaborate)

$$f_{\underset{\sim}{Y}\circ\underset{\sim}{X}}(x,y) = \sup_{\nu} \operatorname{Min}\left[f_{\underset{\sim}{X}}(x,\nu), f_{\underset{\sim}{Y}}(\nu,y)\right] \rightarrow \text{next p}$$

* Relation Example

Recall:

$$\begin{array}{c|cc} & \overset{\nu\rightarrow}{\text{Tom}} & \text{Dick} \\ \hline \text{John } x\downarrow & 0.8 & 0.2 \\ \text{Jim} & 0.6 & 0.9 \end{array} \Bigg\} f_{\underset{\sim}{X}}(x;\nu)$$

Consider Harry & Frank on next page

$$\begin{array}{c|cc} & \overset{y\rightarrow}{\text{Harry}} & \text{Frank} \\ \hline \text{Tom } \nu\downarrow & 0.8 & 0.4 \\ \text{Dick} & 0.1 & 0.6 \end{array} \Bigg\} f_{\underset{\sim}{Y}}(\nu;y)$$

Composition:
Like matrix multipication:

min replaces ×
$\underbrace{\text{max}}_{\text{or sup}}$ replaces +

$$f_{\underset{\sim}{Y}\circ\underset{\sim}{X}}(x,y) \Rightarrow \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.9 \end{bmatrix} \circ \begin{bmatrix} 0.8 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}$$

(Composition Next Page)

max [min (0.8, 0.8), min (0.2, 0.1)] = 0.8
0.8 0.4 0.2 0.6 = 0.4
0.6 0.8 0.9 0.1 = 0.6
0.6 0.4 0.9 0.6 = 0.6

Interpretation:

Resemblance

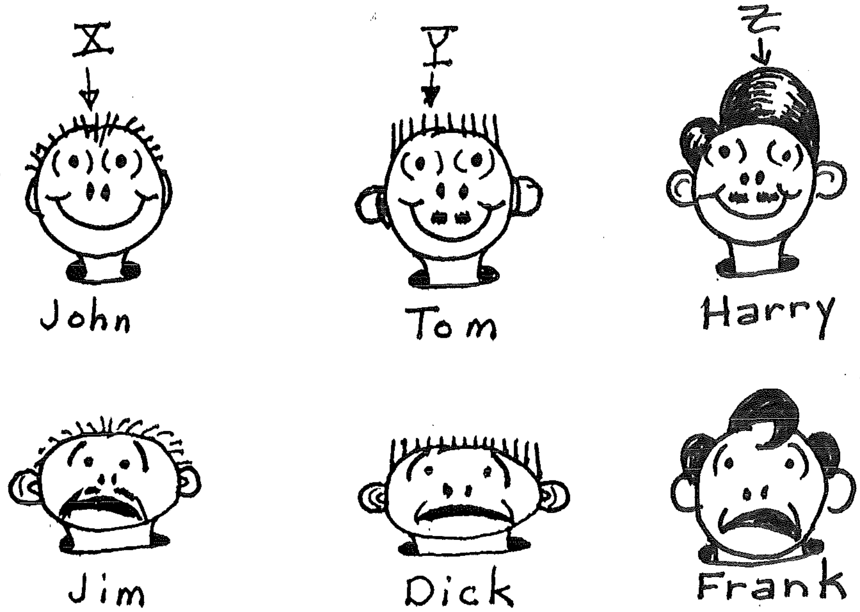
Degree of resemblence of $(x \;, y)$ is
the degree of resemblence of (x, ν_1) <u>and</u> $(\nu_1 y)$
<u>or</u> " " " " " (x, ν_2) <u>"</u> (ν_2, y)
⋮
or " " " " " (x, ν_n) " (ν_n, y)

$$\Rightarrow f_{Y_o \circ X}(x, y)$$

$$= \max_{n} \quad \min \left[f(x, \nu_n), f(\nu_n, y) \right]$$

(This is Discrete Version)

⟵ (return)



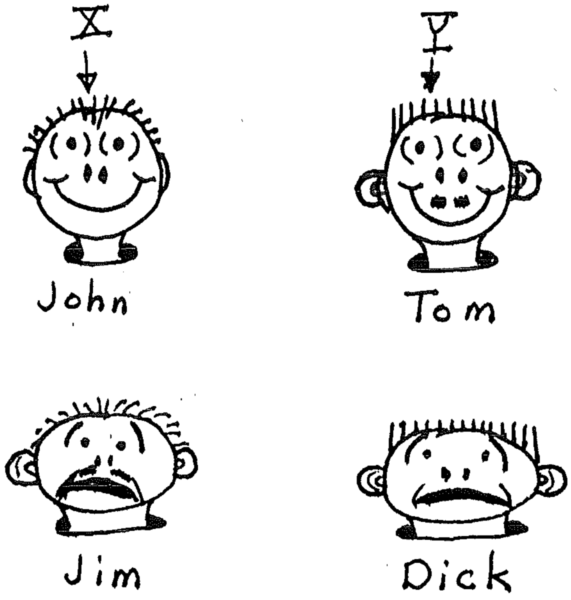
	Harry	Frank
Tom	0.8	0.4
Dick	0.1	0.6

Composition result:

	Harry	Frank
John	0.8	0.4
Jim	0.6	0.6

↑
whiskers!
Ears

Example



$$\begin{array}{cc} \mathbb{X}\times\mathbb{Y} & \begin{array}{cc}\text{John} & \text{Jim}\end{array} \\ \begin{array}{c}\text{Tom}\\ \text{Dick}\end{array} & \underbrace{\begin{array}{cc}0.8 & 0.6\\ 0.2 & 0.9\end{array}}_{\text{Relation Matrix}} \end{array}$$

Properties of Composition

- Associative

$$A \circ (B \circ C) = (A \circ B) \circ C$$

- _not_ Commutative

$A \circ B \Rightarrow B \circ A$, like matrices,
may not even
be defined due
to dimensional
mismatch.

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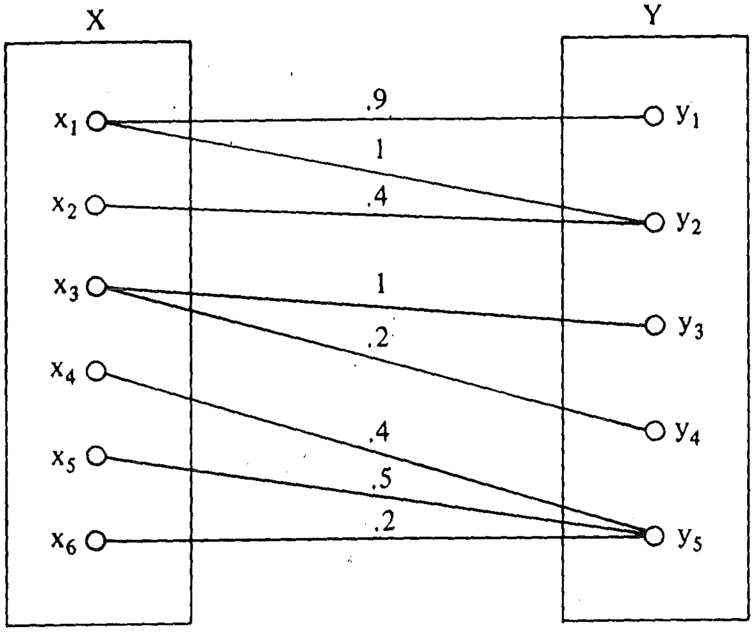


(a)

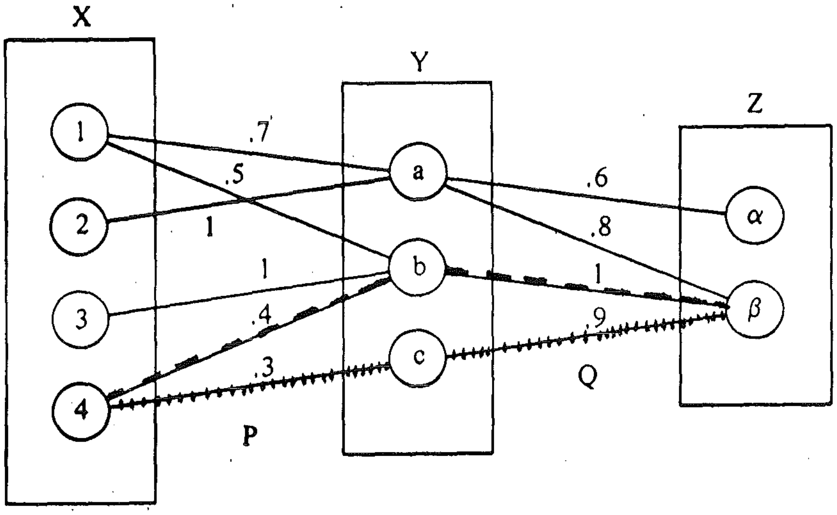
$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} & \left[\begin{array}{ccccc} .9 & 1 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 \\ 0 & 0 & 1 & .2 & 0 \\ 0 & 0 & 0 & 0 & .4 \\ 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & .2 \end{array}\right] \end{array}$$

(b)

Fuzzy relations can also be represented using sagittal diagrams (top). The corresponding relation matrix is shown on the bottom.

(from Klir & Folger, p. 72)

Composition using sagittal diagrams ㉖ (Klir & Folger)



For 4 β:

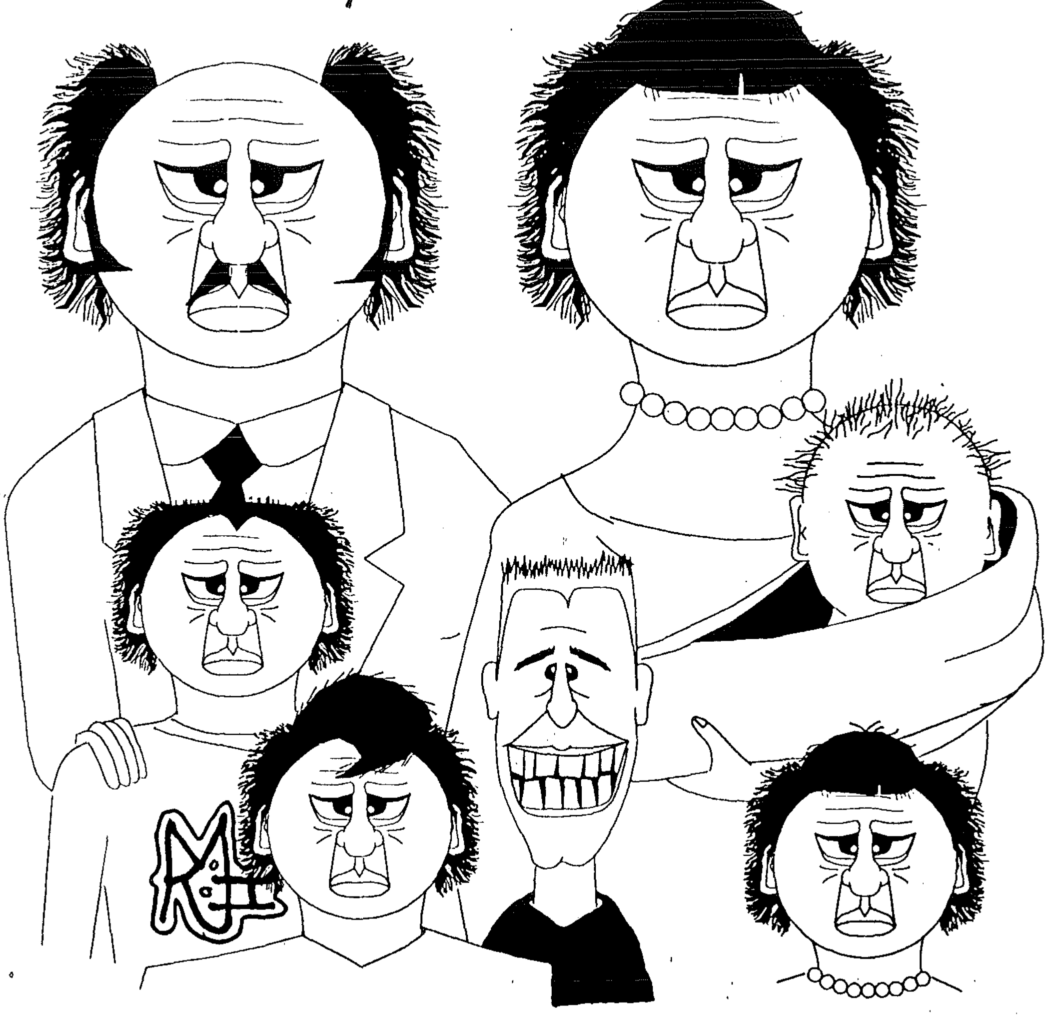
$$\max[\min(0.4, 1), \min(0.3, 0.9)] = 0.4$$

Composition: R = P ∘ Q		
x	z	$\mu_R(x, z)$
1	α	.6
1	β	.7
2	α	.6
2	β	.8
3	β	1
4	β	.4

$$\begin{array}{c|cc} & \alpha & \beta \\ \hline 1 & 0.6 & 0.7 \\ 2 & 0.6 & 0.8 \\ 3 & 0 & 1 \\ 4 & 0 & 0.4 \end{array}$$

㉗

Fuzzy Resemblance



It was later that same year when Peter first suspected that he was adopted.

UNIVERSITY OF WASHINGTON

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April 25, 1994

EE400 Project

There are three options for the project for this course. The emulator project will earn a maximum of fifteen points. The report project can earn a maximum of ten. Innovative projects, the third option, are open ended.

The project will have a total of ten points associated with it. Thus, high scores on an emulator project will count as extra credit.

Material on fuzzy systems will be available from Shinhak Lee during his office hours (9 to 10:30 AM on Mondays and Wednesdays). He will have available

- *Proceedings of the First IEEE International Conference on Fuzzy Systems*, (FUZZ-IEEE), 1992
- *Proceedings of the Second IEEE International Conference on Fuzzy Systems*, (FUZZ-IEEE), 199333
- Bezdek & Pal, a volume containing reprints in classic papers in the application of fuzzy reasoning to pattern recognition.
- Marks, a collection of papers on successful applications of neural networks published in the last few years by IEEE. The volume is not yet in print, but copies of most of the papers in the book are available.

There are three options

Emulator Project

Students choosing this topic will emulate a fuzzy system for a topic of their choosing. Possible applications include fuzzy expert systems, fuzzy control and fuzzy pattern recognition. Students will prepare a written report on their project.

Written and Oral Report

A paper or topic dealing with fuzzy systems will be chosen. The student will write a one page summary of the paper and give an oral presentation of contents of the paper to the class. The length of the presentation depends on the number of students chooing this topic. Fifteen minutes is a good guess. Questions concerning the presentations will be on the final examination.

Open Ended

Innovative projects not falling into the above categories are welcome.

Proposal

Proposals are to be handed in on May 16, 1994. Consideration will be made that week. For the written and oral summary of a paper, include the authors, title, publication, and date of the paper. The written summary is due at the time the oral presentation is given. Those doing emulations should describe their projects in about one page. The final report is due Monday, June 6, 1994. Those doing projects that are publishable will receive a 4.0 in the course.

Robert J. Marks II
Professor of Electrical Engineering

Fuzzy Sets*

L. A. Zadeh

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A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

I. INTRODUCTION

More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1.

Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.

The purpose of this note is to explore in a preliminary way some of the basic properties and implications of a concept which may be of use in dealing with "classes" of the type cited above. The concept in question is that of a *fuzzy set*,[1] that is, a "class" with a continuum of grades of membership. As will be seen in the sequel, the notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.

We begin the discussion of fuzzy sets with several basic definitions.

II. DEFINITIONS

Let X be a space of points (objects), with a generic element of X denoted by x. Thus, $X = \{x\}$.

---

* This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Grant No. AF-AFOSR-139-64 and by the National Science Foundation under Grant GP-2413.

[1] An application of this concept to the formulation of a class of problems in pattern classification is described in RAND Memorandum RM-4307-PR, "Abstraction and Pattern Classification," by R. Bellman, R. Kalaba and L. A. Zadeh, October, 1964.

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A *fuzzy set* (*class*) A in X is characterized by a *membership* (*characteristic*) *function* $f_A(x)$ which associates with each point[2] in X a real number in the interval $[0, 1]$,[3] with the value of $f_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A. When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A. Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set A. (When there is a need to differentiate between such sets and fuzzy sets, the sets with two-valued characteristic functions will be referred to as *ordinary sets* or simply *sets*.)

Example. Let X be the real line R^1 and let A be a fuzzy set of numbers which are much greater than 1. Then, one can give a precise, albeit subjective, characterization of A by specifying $f_A(x)$ as a function on R^1. Representative values of such a function might be: $f_A(0) = 0$; $f_A(1) = 0$; $f_A(5) = 0.01$; $f_A(10) = 0.2$; $f_A(100) = 0.95$; $f_A(500) = 1$.

It should be noted that, although the membership function of a fuzzy set has some resemblance to a probability function when X is a countable set (or a probability density function when X is a continuum), there are essential differences between these concepts which will become clearer in the sequel once the rules of combination of membership functions and their basic properties have been established. In fact, the notion of a fuzzy set is completely nonstatistical in nature.

We begin with several definitions involving fuzzy sets which are obvious extensions of the corresponding definitions for ordinary sets.

A fuzzy set is *empty* if and only if its membership function is identically zero on X.

Two fuzzy sets A and B are *equal*, written as $A = B$, if and only if $f_A(x) = f_B(x)$ for all x in X. (In the sequel, instead of writing $f_A(x) = f_B(x)$ for all x in X, we shall write more simply $f_A = f_B$.)

The *complement* of a fuzzy set A is denoted by A' and is defined by

$$f_{A'} = 1 - f_A . \tag{1}$$

As in the case of ordinary sets, the notion of containment plays a central role in the case of fuzzy sets. This notion and the related notions of union and intersection are defined as follows.

Containment. A is *contained in* B (or, equivalently, A is a *subset of* B, or A is *smaller than or equal to* B) if and only if $f_A \leqq f_B$. In symbols

$$A \subset B \Leftrightarrow f_A \leqq f_B . \tag{2}$$

Union. The *union* of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to those of A and B by

$$f_C(x) = \mathrm{Max}\,[f_A(x), f_B(x)], \qquad x \in X \tag{3}$$

or, in abbreviated form

$$f_C = f_A \vee f_B . \tag{4}$$

Note that $\cup$ has the associative property, that is, $A \cup (B \cup C) = (A \cup B) \cup C$.

Comment. A more intuitively appealing way of defining the union is

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[2] More generally, the domain of definition of $f_A(x)$ may be restricted to a subset of X.

[3] In a more general setting, the range of the membership function can be taken to be a suitable partially ordered set P. For our purposes, it is convenient and sufficient to restrict the range of f to the unit interval. If the values of $f_A(x)$ are interpreted as truth values, the latter case corresponds to a multivalued logic with a continuum of truth values in the interval $[0, 1]$.

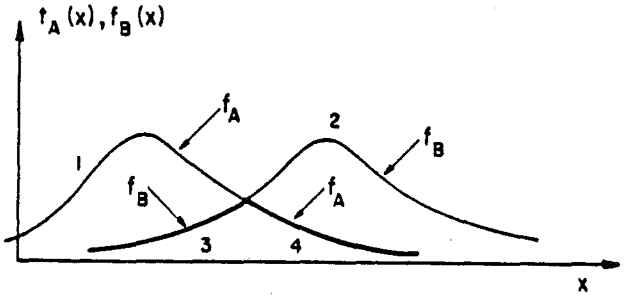

FIG. 1. Illustration of the union and intersection of fuzzy sets in R^1

the following: The union of A and B is the smallest fuzzy set containing both A and B. More precisely, if D is any fuzzy set which contains both A and B, then it also contains the union of A and B.

To show that this definition is equivalent to (3), we note, first, that C as defined by (3) contains both A and B, since

$$\text{Max}\,[f_A, f_B] \geqq f_A$$

and

$$\text{Max}\,[f_A, f_B] \geqq f_B .$$

Furthermore, if D is any fuzzy set containing both A and B, then

$$f_D \geqq f_A$$

$$f_D \geqq f_B$$

and hence

$$f_D \geqq \text{Max}\,[f_A, f_B] = f_C$$

which implies that $C \subset D$. Q.E.D.

The notion of an intersection of fuzzy sets can be defined in an analogous manner. Specifically:

Intersection. The *intersection* of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to those of A and B by

$$f_C(x) = \text{Min}\,[f_A(x), f_B(x)], \qquad x \in X, \tag{5}$$

or, in abbreviated form

$$f_C = f_A \wedge f_B . \tag{6}$$

As in the case of the union, it is easy to show that the intersection of A and B is the *largest* fuzzy set which is contained in both A and B. As in the case of ordinary sets, A and B are *disjoint* if $A \cap B$ is empty. Note that $\cap$, like $\cup$, has the associative property.

The intersection and union of two fuzzy sets in R^1 are illustrated in Fig. 1. The membership function of the union is comprised of curve segments 1 and 2; that of the intersection is comprised of segments 3 and 4 (heavy lines).

Comment. Note that the notion of "belonging," which plays a fundamental role in the case of ordinary sets, does not have the same role in the case of fuzzy sets. Thus, it is not meaningful to speak of a point x "belonging" to a fuzzy set A except in the trivial sense of $f_A(x)$ being positive. Less trivially, one can introduce two levels α and β ($0 < \alpha < 1$, $0 < \beta < 1$, $\alpha > \beta$) and agree to say that (1) "x belongs to A" if $f_A(x) \geqq \alpha$; (2) "x does not belong to A" if $f_A(x) \leqq \beta$; and (3) "x has an indeterminate status relative to A" if $\beta < f_A(x) < \alpha$. This leads to a three-valued logic (Kleene, 1952) with three truth values: T $(f_A(x) \geqq \alpha)$, F $(f_A(x) \leqq \beta)$, and U $(\beta < f_A(x) < \alpha)$.

III. SOME PROPERTIES OF $\cup$, $\cap$, AND COMPLEMENTATION

With the operations of union, intersection, and complementation defined as in (3), (5), and (1), it is easy to extend many of the basic identities which hold for ordinary sets to fuzzy sets. As examples, we have

$$\begin{array}{ll}\left.\begin{array}{l}(A \cup B)' = A' \cap B' \\ (A \cap B)' = A' \cup B'\end{array}\right\}\text{De Morgan's laws} & \begin{array}{r}(7)\\(8)\end{array}\end{array}$$

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B) \qquad \text{Distributive laws.} \quad (9)$$

$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B) \quad (10)$$

These and similar equalities can readily be established by showing that the corresponding relations for the membership functions of A, B, and C are identities. For example, in the case of (7), we have

$$1 - \text{Max}\,[f_A\,, f_B] = \text{Min}\,[1 - f_A\,, 1 - f_B] \quad (11)$$

which can be easily verified to be an identity by testing it for the two possible cases: $f_A(x) > f_B(x)$ and $f_A(x) < f_B(x)$.

Similarly, in the case of (10), the corresponding relation in terms of f_A, f_B, and f_C is:

$$\text{Max}\,[f_C\,, \text{Min}\,[f_A\,, f_B] = \text{Min}\,[\text{Max}\,[f_C\,, f_A], \text{Max}\,[f_C\,, f_B]] \quad (12)$$

which can be verified to be an identity by considering the six cases:

$$f_A(x) > f_B(x) > f_C(x), f_A(x) > f_C(x) > f_B(x), f_B(x) > f_A(x) > f_C(x),$$

$$f_B(x) > f_C(x) > f_A(x), f_C(x) > f_A(x) > f_B(x), f_C(x) > f_B(x) > f_A(x).$$

Essentially, fuzzy sets in X constitute a distributive lattice with a 0 and 1 (Birkhoff, 1948).

AN INTERPRETATION FOR UNIONS AND INTERSECTIONS

In the case of ordinary sets, a set C which is expressed in terms of a family of sets $A_1, \cdots, A_i, \cdots, A_n$ through the connectives $\cup$ and $\cap$, can be represented as a network of switches $\alpha_1, \cdots, \alpha_n$, with $A_i \cap A_j$ and $A_i \cup A_j$ corresponding, respectively, to series and parallel combinations of α_i and α_j. In the case of fuzzy sets, one can give an analogous interpretation in terms of sieves. Specifically, let $f_i(x)$, $i = 1, \cdots, n$, denote the value of the membership function of A_i at x. Associate with $f_i(x)$ a sieve $S_i(x)$ whose meshes are of size $f_i(x)$. Then, $f_i(x) \vee f_j(x)$ and $f_i(x) \wedge f_j(x)$ correspond, respectively, to parallel and series combinations of $S_i(x)$ and $S_j(x)$, as shown in Fig. 2.

More generally, a well-formed expression involving $A_1, \cdots, A_n$, $\cup$, and $\cap$ corresponds to a network of sieves $S_1(x), \cdots, S_n(x)$ which can be found by the conventional synthesis techniques for switching circuits. As a very simple example,

$$C = [(A_1 \cup A_2) \cap A_3] \cup A_4 \quad (13)$$

corresponds to the network shown in Fig. 3.

Note that the mesh sizes of the sieves in the network depend on x and that the network as a whole is equivalent to a single sieve whose meshes are of size $f_C(x)$.

IV. ALGEBRAIC OPERATIONS ON FUZZY SETS

In addition to the operations of union and intersection, one can define a number of other ways of forming combinations of fuzzy sets and relating them to one another. Among the more important of these are the following.

Algebraic product. The *algebraic product* of A and B is denoted by AB

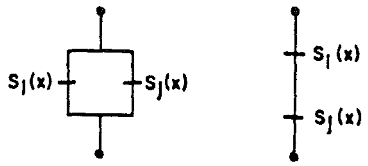

FIG. 2. Parallel and series connection of sieves simulating $\cup$ and $\cap$

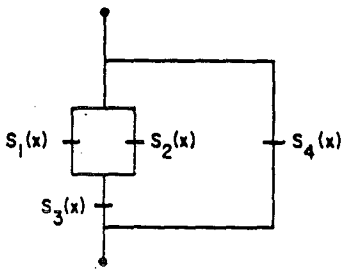

FIG. 3. A network of sieves simulating $\{[f_1(x) \vee f_2(x)] \wedge f_3(x)\} \vee f_4(x)$

and is defined in terms of the membership functions of A and B by the relation

$$f_{AB} = f_A f_B . \tag{14}$$

Clearly,

$$AB \subset A \cap B. \tag{15}$$

Algebraic sum.[4] The *algebraic sum* of A and B is denoted by $A + B$ and is defined by

$$f_{A+B} = f_A + f_B \tag{16}$$

provided the sum $f_A + f_B$ is less than or equal to unity. Thus, unlike the algebraic product, the algebraic sum is meaningful only when the condition $f_A(x) + f_B(x) \leqq 1$ is satisfied for all x.

Absolute difference. The *absolute difference* of A and B is denoted by $|A - B|$ and is defined by

$$f_{|A-B|} = |f_A - f_B|.$$

Note that in the case of ordinary sets $|A - B|$ reduces to the relative complement of $A \cap B$ in $A \cup B$.

Convex combination. By a convex combination of two vectors f and g is usually meant a linear combination of f and g of the form $\lambda f + (1 - \lambda)g$, in which $0 \leqq \lambda \leqq 1$. This mode of combining f and g can be generalized to fuzzy sets in the following manner.

Let A, B, and Λ be arbitrary fuzzy sets. The *convex combination of A, B, and Λ* is denoted by $(A, B; \Lambda)$ and is defined by the relation

$$(A, B; \Lambda) = \Lambda A + \Lambda' B \tag{17}$$

where Λ' is the complement of Λ. Written out in terms of membership functions, (17) reads

$$f_{(A,B;\Lambda)}(x) = f_\Lambda(x) f_A(x) + [1 - f_\Lambda(x)] f_B(x), \qquad x \in X. \tag{18}$$

A basic property of the convex combination of A, B, and Λ is expressed by

$$A \cap B \subset (A, B; \Lambda) \subset A \cup B \qquad \text{for all } \Lambda. \tag{19}$$

---

[4] The dual of the algebraic product is the *sum* $A \oplus B = (A'B')' = A + B - AB$. (This was pointed out by T. Cover.) Note that for ordinary sets $\cap$ and the algebraic product are equivalent operations, as are $\cup$ and $\oplus$.

This property is an immediate consequence of the inequalities

$$\text{Min}\,[f_A(x), f_B(x)] \leqq \lambda f_A(x) + (1 - \lambda) f_B(x) \leqq \text{Max}\,[f_A(x), f_B(x)], \qquad x \in X \tag{20}$$

which hold for all λ in $[0, 1]$. It is of interest to observe that, given any fuzzy set C satisfying $A \cap B \subset C \subset A \cup B$, one can always find a fuzzy set Λ such that $C = (A, B; \Lambda)$. The membership function of this set is given by

$$f_\Lambda(x) = \frac{f_C(x) - f_B(x)}{f_A(x) - f_B(x)}, \qquad x \in X. \tag{21}$$

Fuzzy relation. The concept of a *relation* (which is a generalization of that of a *function*) has a natural extension to fuzzy sets and plays an important role in the theory of such sets and their applications—just as it does in the case of ordinary sets. In the sequel, we shall merely define the notion of a fuzzy relation and touch upon a few related concepts.

Ordinarily, a relation is defined as a set of ordered pairs (Halmos, 1960); e.g., the set of all ordered pairs of real numbers x and y such that $x \geqq y$. In the context of fuzzy sets, a *fuzzy relation in* X is a fuzzy set in the product space $X \times X$. For example, the relation denoted by $x \gg y$, $x, y \in R^1$, may be regarded as a fuzzy set A in R^2, with the membership function of A, $f_A(x, y)$, having the following (subjective) representative values: $f_A(10, 5) = 0$; $f_A(100, 10) = 0.7$; $f_A(100, 1) = 1$; etc.

More generally, one can define an *n-ary fuzzy relation* in X as a fuzzy set A in the product space $X \times X \times \cdots \times X$. For such relations, the membership function is of the form $f_A(x_1, \cdots, x_n)$, where $x_i \in X$, $i = 1, \cdots, n$.

In the case of binary fuzzy relations, the *composition* of two fuzzy relations A and B is denoted by $B \circ A$ and is defined as a fuzzy relation in X whose membership function is related to those of A and B by

$$f_{B \circ A}(x, y) = \text{Sup}_v\, \text{Min}\,[f_A(x, v), f_B(v, y)].$$

Note that the operation of composition has the associative property

$$A \circ (B \circ C) = (A \circ B) \circ C.$$

Fuzzy sets induced by mappings. Let T be a mapping from X to a space Y. Let B be a fuzzy set in Y with membership function $f_B(y)$. The inverse mapping T^{-1} induces a fuzzy set A in X whose membership function is defined by

$$f_A(x) = f_B(y), \qquad y \in Y \tag{22}$$

for all x in X which are mapped by T into y.

Consider now a converse problem in which A is a given fuzzy set in X, and T, as before, is a mapping from X to Y. The question is: What is the membership function for the fuzzy set B in Y which is induced by this mapping?

If T is not one-one, then an ambiguity arises when two or more distinct points in X, say x_1 and x_2, with different grades of membership in A, are mapped into the same point y in Y. In this case, the question is: What grade of membership in B should be assigned to y?

To resolve this ambiguity, we agree to assign the larger of the two grades of membership to y. More generally, the membership function for B will be defined by

$$f_B(y) = \text{Max}_{x \in T^{-1}(y)} f_A(x), \qquad y \in Y \tag{23}$$

where $T^{-1}(y)$ is the set of points in X which are mapped into y by T.

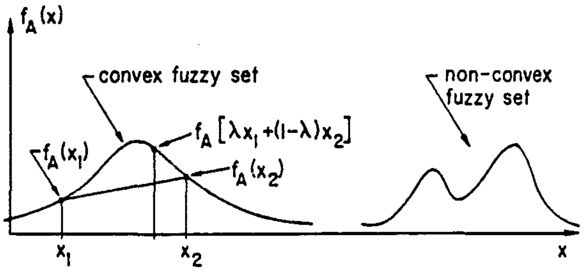

FIG. 4. Convex and nonconvex fuzzy sets in E^1

V. CONVEXITY

As will be seen in the sequel, the notion of convexity can readily be extended to fuzzy sets in such a way as to preserve many of the properties which it has in the context of ordinary sets. This notion appears to be particularly useful in applications involving pattern classification, optimization and related problems.

In what follows, we assume for concreteness that X is a real Euclidean space E^n.

DEFINITIONS

Convexity. A fuzzy set A is *convex* if and only if the sets Γ_α defined by

$$\Gamma_\alpha = \{x \mid f_A(x) \geqq \alpha\} \tag{24}$$

are convex for all α in the interval (0, 1].

An alternative and more direct definition of convexity is the following[5]: A is *convex* if and only if

$$f_A[\lambda x_1 + (1 - \lambda)x_2] \geqq \text{Min}\,[f_A(x_1), f_A(x_2)] \tag{25}$$

for all x_1 and x_2 in X and all λ in [0, 1]. Note that this definition does not imply that $f_A(x)$ must be a convex function of x. This is illustrated in Fig. 4 for $n = 1$.

To show the equivalence between the above definitions note that if A is convex in the sense of the first definition and $\alpha = f_A(x_1) \leqq f_A(x_2)$, then $x_2 \in \Gamma_\alpha$ and $\lambda x_1 + (1 - \lambda)x_2 \in \Gamma_\alpha$ by the convexity of Γ_α. Hence

$$f_A[\lambda x_1 + (1 - \lambda)x_2] \geqq \alpha = f_A(x_1) = \text{Min}\,[f_A(x_1), f_A(x_2)].$$

Conversely, if A is convex in the sense of the second definition and $\alpha = f_A(x_1)$, then Γ_α may be regarded as the set of all points x_2 for which $f_A(x_2) \geqq f_A(x_1)$. In virtue of (25), every point of the form $\lambda x_1 + (1 - \lambda)x_2$, $0 \leqq \lambda \leqq 1$, is also in Γ_α and hence Γ_α is a convex set. Q.E.D.

A basic property of convex fuzzy sets is expressed by the

THEOREM. *If A and B are convex, so is their intersection.*

Proof: Let $C = A \cap B$. Then

$$\begin{aligned} &f_C[\lambda x_1 + (1 - \lambda)x_2] \\ &\qquad = \text{Min}\,[f_A[\lambda x_1 + (1 - \lambda)x_2], f_B[\lambda x_1 + (1 - \lambda)x_2]]. \end{aligned} \tag{26}$$

Now, since A and B are convex

$$\begin{aligned} f_A[\lambda x_1 + (1 - \lambda)x_2] &\geqq \text{Min}\,[f_A(x_1), f_A(x_2)] \\ f_B[\lambda x_1 + (1 - \lambda)x_2] &\geqq \text{Min}\,[f_B(x_1), f_B(x_2)] \end{aligned} \tag{27}$$

and hence

$$\begin{aligned} &f_C[\lambda x_1 + (1 - \lambda)x_2] \\ &\qquad \geqq \text{Min}\,[\text{Min}\,[f_A(x_1), f_A(x_2)], \text{Min}\,[f_B(x_1), f_B(x_2)]] \end{aligned} \tag{28}$$

---

[5] This way of expressing convexity was suggested to the writer by his colleague, E. Berlekamp.

or equivalently

$$f_C[\lambda x_1 + (1 - \lambda)x_2] \geqq \text{Min}\,[\text{Min}\,[f_A(x_1), f_B(x_1)], \text{Min}\,[f_A(x_2), f_B(x_2)]] \tag{29}$$

and thus

$$f_C[\lambda x_1 + (1 - \lambda)x_2] \geqq \text{Min}\,[f_C(x_1), f_C(x_2)]. \quad \text{Q. E. D.} \tag{30}$$

Boundedness. A fuzzy set A is *bounded* if and only if the sets $\Gamma_\alpha = \{x \mid f_A(x) \geqq \alpha\}$ are bounded for all $\alpha > 0$; that is, for every $\alpha > 0$ there exists a finite $R(\alpha)$ such that $\| x \| \leqq R(\alpha)$ for all x in Γ_α.

If A is a bounded set, then for each $\epsilon > 0$ then exists a hyperplane H such that $f_A(x) \leqq \epsilon$ for all x on the side of H which does not contain the origin. For, consider the set $\Gamma_\epsilon = \{x \mid f_A(x) \geqq \epsilon\}$. By hypothesis, this set is contained in a sphere S of radius $R(\epsilon)$. Let H be any hyperplane supporting S. Then, all points on the side of H which does not contain the origin lie outside or on S, and hence for all such points $f_A(x) \leqq \epsilon$.

LEMMA. *Let A be a bounded fuzzy set and let $M = \text{Sup}_x f_A(x)$. (M will be referred to as the* maximal grade *in A.) Then there is at least one point x_0 at which M is essentially attained in the sense that, for each $\epsilon > 0$, every spherical neighborhood of x_0 contains points in the set $Q(\epsilon) = \{x \mid f_A(x) \geqq M - \epsilon\}$.*

Proof.[6] Consider a nested sequence of bounded sets $\Gamma_1, \Gamma_2, \cdots$, where $\Gamma_n = \{x \mid f_A(x) \geqq M - M/(n + 1)\}$, $n = 1, 2, \cdots$. Note that Γ_n is nonempty for all finite n as a consequence of the definition of M as $M = \text{Sup}_x f_A(x)$. (We assume that $M > 0$.)

Let x_n be an arbitrarily chosen point in Γ_n, $n = 1, 2, \cdots$. Then, $x_1, x_2, \cdots$, is a sequence of points in a closed bounded set Γ_1. By the Bolzano-Weierstrass theorem, this sequence must have at least one limit point, say x_0, in Γ_1. Consequently, every spherical neighborhood of x_0 will contain infinitely many points from the sequence $x_1, x_2, \cdots$, and, more particularly, from the subsequence $x_{N+1}, x_{N+2}, \cdots$, where $N \geqq M/\epsilon$. Since the points of this subsequence fall within the set $Q(\epsilon) = \{x \mid f_A(x) \geqq M - \epsilon\}$, the lemma is proved.

Strict and strong convexity. A fuzzy set A is *strictly convex* if the sets Γ_α, $0 < \alpha \leqq 1$ are strictly convex (that is, if the midpoint of any two distinct points in Γ_α lies in the interior of Γ_α). Note that this definition reduces to that of strict convexity for ordinary sets when A is such a set.

A fuzzy set A is *strongly convex* if, for any two distinct points x_1 and x_2, and any λ in the open interval $(0, 1)$

$$f_A[\lambda x_1 + (1 - \lambda)x_2] > \text{Min}\,[f_A(x_1), f_A(x_2)].$$

Note that strong convexity does not imply strict convexity or vice-versa. Note also that if A and B are bounded, so is their union and intersection. Similarly, if A and B are strictly (strongly) convex, their intersection is strictly (strongly) convex.

Let A be a convex fuzzy set and let $M = \text{Sup}_x f_A(x)$. If A is bounded, then, as shown above, either M is attained for some x, say x_0, or there is at least one point x_0 at which M is essentially attained in the sense that, for each $\epsilon > 0$, every spherical neighborhood of x_0 contains points in the set $Q(\epsilon) = \{x \mid M - f_A(x) \leqq \epsilon\}$. In particular, if A is strongly convex and x_0 is attained, then x_0 is unique. For, if $M = f_A(x_0)$ and $M = f_A(x_1)$, with $x_1 \neq x_0$, then $f_A(x) > M$ for $x = 0.5x_0 + 0.5x_1$, which contradicts $M = \text{Max}_x f_A(x)$.

More generally, let $C(A)$ be the set of all points in X at which M is essentially attained. This set will be referred to as the *core* of A. In the case of convex fuzzy sets, we can assert the following property of $C(A)$.

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[6] This proof was suggested by A. J. Thomasian.

THEOREM. *If A is a convex fuzzy set, then its core is a convex set.*

Proof: It will suffice to show that if M is essentially attained at x_0 and x_1, $x_1 \neq x_0$, then it is also essentially attained at all x of the form $x = \lambda x_0 + (1 - \lambda)x_1$, $0 \leqq \lambda \leqq 1$.

To the end, let P be a cylinder of radius ϵ with the line passing through x_0 and x_1 as its axis. Let x_0' be a point in a sphere of radius ϵ centering on x_0 and x_1' be a point in a sphere of radius ϵ centering on x_1 such that $f_A(x_0') \geqq M - \epsilon$ and $f_A(x_1') \geqq M - \epsilon$. Then, by the convexity of A, for any point u on the segment $x_0'x_1'$, we have $f_A(u) \geqq M - \epsilon$. Furthermore, by the convexity of P, all points on $x_0'x_1'$ will lie in P.

Now let x be any point in the segment x_0x_1. The distance of this point from the segment $x_0'x_1'$ must be less than or equal to ϵ, since $x_0'x_1'$ lies in P. Consequently, a sphere of radius ϵ centering on x will contain at least one point of the segment $x_0'x_1'$ and hence will contain at least one point, say w, at which $f_A(w) \geqq M - \epsilon$. This establishes that M is essentially attained at x and thus proves the theorem.

COROLLARY. *If $X = E^1$ and A is strongly convex, then the point at which M is essentially attained is unique.*

Shadow of a fuzzy set. Let A be a fuzzy set in E^n with membership function $f_A(x) = f_A(x_1, \cdots, x_n)$. For notational simplicity, the notion of the *shadow* (projection) of A on a hyperplane H will be defined below for the special case where H is a coordinate hyperplane, e.g., $H = \{x \mid x_1 = 0\}$.

Specifically, the *shadow* of A on $H = \{x \mid x_1 = 0\}$ is defined to be a fuzzy set $S_H(A)$ in E^{n-1} with $f_{S_H(A)}(x)$ given by

$$f_{S_H(A)}(x) = f_{S_H(A)}(x_2, \cdots, x_n) = \text{Sup}_{x_1} f_A(x_1, \cdots, x_n).$$

Note that this definition is consistent with (23).

When A is a convex fuzzy set, the following property of $S_H(A)$ is an immediate consequence of the above definition: If A is a convex fuzzy set, then its shadow on any hyperplane is also a convex fuzzy set.

An interesting property of the shadows of two convex fuzzy sets is expressed by the following implication

$$S_H(A) = S_H(B) \text{ for all } H \Rightarrow A = B.$$

To prove this assertion,[7] it is sufficient to show that if there exists a point, say x_0, such that $f_A(x_0) \neq f_B(x_0)$, then their exists a hyperplane H such that $f_{S_H(A)}(x_0^*) \neq f_{S_H(B)}(x_0^*)$, where x_0^* is the projection of x_0 on H.

Suppose that $f_A(x_0) = \alpha > f_B(x_0) = \beta$. Since B is a convex fuzzy set, the set $\Gamma_\beta = \{x \mid f_B(x) > \beta\}$ is convex, and hence there exists a hyperplane F supporting Γ_β and passing through x_0. Let H be a hyperplane orthogonal to F, and let x_0^* be the projection of x_0 on H. Then, since $f_B(x) \leqq \beta$ for all x on F, we have $f_{S_H(B)}(x_0^*) \leqq \beta$. On the other hand, $f_{S_H(A)}(x_0^*) \geqq \alpha$. Consequently, $f_{S_H(B)}(x_0^*) \neq f_{S_H(A)}(x_0^*)$, and similarly for the case where $\alpha < \beta$.

A somewhat more general form of the above assertion is the following: Let A, but not necessarily B, be a convex fuzzy set, and let $S_H(A) = S_H(B)$ for all H. Then $A = \text{conv } B$, where conv B is the convex hull of B, that is, the smallest convex set containing B. More generally, $S_H(A) = S_H(B)$ for all H implies conv A = conv B.

Separation of convex fuzzy sets. The classical separation theorem for ordinary convex sets states, in essence, that if A and B are disjoint convex sets, then there exists a separating hyperplane H such that A is on one side of H and B is on the other side.

It is natural to inquire if this theorem can be extended to convex fuzzy

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[7] This proof is based on an idea suggested by G. Dantzig for the case where A and B are ordinary convex sets.

sets, without requiring that A and B be disjoint, since the condition of disjointness is much too restrictive in the case of fuzzy sets. It turns out, as will be seen in the sequel, that the answer to this question is in the affirmative.

As a preliminary, we shall have to make a few definitions. Specifically, let A and B be two bounded fuzzy sets and let H be a hypersurface in E^n defined by an equation $h(x) = 0$, with all points for which $h(x) \geqq 0$ being on one side of H and all points for which $h(x) \leqq 0$ being on the other side.[8] Let K_H be a number dependent on H such that $f_A(x) \leqq K_H$ on one side of H and $f_B(x) \leqq K_H$ on the other side. Let M_H be Inf K_H. The number $D_H = 1 - M_H$ will be called the *degree of separation of A and B by H.*

In general, one is concerned not with a given hypersurface H, but with a family of hypersurfaces $\{H_\lambda\}$, with λ ranging over, say, E^m. The problem, then, is to find a member of this family which realizes the highest possible degree of separation.

A special case of this problem is one where the H_λ are hyperplanes in E^n, with λ ranging over E^n. In this case, we define the *degree of separability* of A and B by the relation

$$D = 1 - \bar{M} \tag{31}$$

where

$$\bar{M} = \mathrm{Inf}_H M_H \tag{32}$$

with the subscript λ omitted for simplicity.

Among the various assertions that can be made concerning D, the following statement[9] is, in effect, an extension of the separation theorem to convex fuzzy sets.

THEOREM. *Let A and B be bounded convex fuzzy sets in E^n, with maximal grades M_A and M_B, respectively* [$M_A = \mathrm{Sup}_x f_A(x)$, $M_B = \mathrm{Sup}_x f_B(x)$]. *Let M be the maximal grade for the intersection $A \cap B$* ($M = \mathrm{Sup}_x \mathrm{Min} \cdot [f_A(x), f_B(x)]$). *Then $D = 1 - M$.*

Comment. In plain words, the theorem states that the highest degree of separation of two convex fuzzy sets A and B that can be achieved with a hyperplane in E^n is one minus the maximal grade in the intersection $A \cap B$. This is illustrated in Fig. 5 for $n = 1$.

Proof: It is convenient to consider separately the following two cases: (1) $M = \mathrm{Min}\ (M_A, M_B)$ and (2) $M < \mathrm{Min}\ (M_A, M_B)$. Note that the latter case rules out $A \subset B$ or $B \subset A$.

Case 1. For concreteness, assume that $M_A < M_B$, so that $M = M_A$. Then, by the property of bounded sets already stated there exists a hyperplane H such that $f_B(x) \leqq M$ for all x on one side of H. On the other side of H, $f_A(x) \leqq M$ because $f_A(x) \leqq M_A = M$ for all x.

It remains to be shown that there do not exist an $M' < M$ and a hyperplane H' such that $f_A(x) \leqq M'$ on one side of H' and $f_B(x) \leqq M'$ on the other side.

This follows at once from the following observation. Suppose that such H' and M' exist, and assume for concreteness that the core of A (that is, the set of points at which $M_A = M$ is essentially attained) is on the plus side of H'. This rules out the possibility that $f_A(x) \leqq M'$ for all x on the plus side of H', and hence necessitates that $f_A(x) \leqq M'$ for all x on the minus side of H', and $f_B(x) \leqq M'$ for all x on the plus side of H'. Consequently, over all x on the plus side of H'

$$\mathrm{Sup}_x \mathrm{Min}\ [f_A(x), f_B(x)] \leqq M'$$

and likewise for all x on the minus side of H'. This implies that, over all

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[8] Note that the sets in question have H in common.

[9] This statement is based on a suggestion of E. Berlekamp.

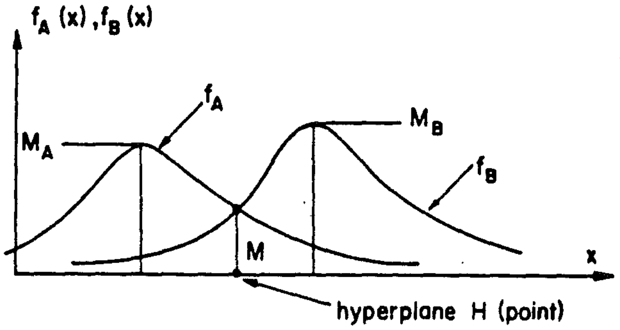

FIG. 5. Illustration of the separation theorem for fuzzy sets in E^1

x in X, $\text{Sup}_x \text{Min}\,[f_A(x), f_B(x)] \leqq M'$, which contradicts the assumption that $\text{Sup}_x \text{Min}\,[f_A(x), f_B(x)] = M > M'$.

Case 2. Consider the convex sets $\Gamma_A = \{x \mid f_A(x) > M\}$ and $\Gamma_B = \{x \mid f_B(x) > M\}$. These sets are nonempty and disjoint, for if they were not there would be a point, say u, such that $f_A(u) > M$ and $f_B(u) > M$, and hence $f_{A \cap B}(u) > M$, which contradicts the assumption that $M = \text{Sup}_x f_{A \cap B}(x)$.

Since Γ_A and Γ_B are disjoint, by the separation theorem for ordinary convex sets there exists a hyperplane H such that Γ_A is on one side of H (say, the plus side) and Γ_B is on the other side (the minus side). Furthermore, by the definitions of Γ_A and Γ_B, for all points on the minus side of $H, f_A(x) \leqq M$, and for all points on the plus side of $H, f_B(x) \leqq M$.

Thus, we have shown that there exists a hyperplane H which realizes $1 - M$ as the degree of separation of A and B. The conclusion that a higher degree of separation of A and B cannot be realized follows from the argument given in Case 1. This concludes the proof of the theorem.

The separation theorem for convex fuzzy sets appears to be of particular relevance to the problem of pattern discrimination. Its application to this class of problems as well as to problems of optimization will be explored in subsequent notes on fuzzy sets and their properties.

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REFERENCES

BIRKHOFF, G. (1948), "Lattice Theory," Am. Math. Soc. Colloq. Publ., Vol. 25, New York.

HALMOS, P. R. (1960), "Naive Set Theory." Van Nostrand, New York.

KLEENE, S. C. (1952), "Introduction to Metamathematics," p. 334. Van Nostrand, New York.

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Fuzzy Systems on TIE

Spring QUARTER 1994

EE400: ***Introduction to Fuzzy Systems***

Prof. Robert J. Marks II

Three Credits

Spring quarter, 1994; 8:30 to 10:00 Tues/Thurs.

Prerequisite: EE370, STAT390 or permission of the instructor.

- Texts:
  - ♦ G.J. Klir and T.A. Folger, ***Fuzzy Sets, Uncertainty and Information***, Prentice Hall, 1988.
  - ♦ R.J. Marks II, Editor, **Fuzzy Logic Technology and Applications I**, (IEEE Technical Activities Board, Piscataway, 1994).

- Reference:
  - ♦ T. Terano, K. Asai & M.Sugeno, ***Fuzzy Systems Theory and Its Applications***, Academic Press, 1992
  - ♦ J.C. Bezdek & S.K. Pal, ***Fuzzy Models for Pattern Recognition***, IEEE Press, 1992.
  - ♦ *Proceedings of FUZZ-IEEE*
  - ♦ *IEEE Transactions on Fuzzy Systems*

- Course Outline
  - ♦ Crisp sets and fuzzy sets.
  - ♦ Operations on Fuzzy sets
  - ♦ Fuzzy relations.
  - ♦ Fuzzy measures.
  - ♦ Adaptive Fuzzy Processing.
  - ♦ Uncertainty and information.
  - ♦ Applications.

ABSTRACT:

Fuzzy theory, in part, seeks to provide an accurate and useful model of uncertainty. Fuzzy modeling, based on relative membership of elements in sets, has a firm axiomatic base from which advance concepts are developed. These concepts are then applied to such problems as fuzzy inferencing, fuzzy control and fuzzy pattern recognition. Prerequisites include a firm working knowledge of both conventional (crisp) set theory and probability.

- Students will perform a course project.

Engineering science: 2 credits
Engineering design: 1 credits

SUMMER QUARTER 1992

EE595: *Introduction to Fuzzy Systems*

Prof. Robert J. Marks II

Three Credits

Summer quarter, 1992.

Prerequisite: EE505

- Texts:
    - G.J. Klir and T.A. Folger, ***Fuzzy Sets, Uncertainty and Information*** , Prentice Hall, 1988.
    - T. Terano, K. Asai & M.Sugeno, ***Fuzzy Systems Theory and Its Applications*** , Academic Press, 1992

- Reference:
    - J.C. Bezdek & S.K. Pal, ***Fuzzy Models for Pattern Recognition*** , IEEE Press, 1992.
    - *'92 FUZZ-IEEE Proceedings*

- Course Outline
    - Crisp sets and fuzzy sets.
    - Operations on Fuzzy sets
    - Fuzzy relations.
    - Fuzzy measures.
    - Adaptive Fuzzy Processing.
    - Uncertainty and information.
    - Applications.

- Students will read and summarize a paper on a contemporary fuzzy topic.

Enrollment will be limited to 30 students.

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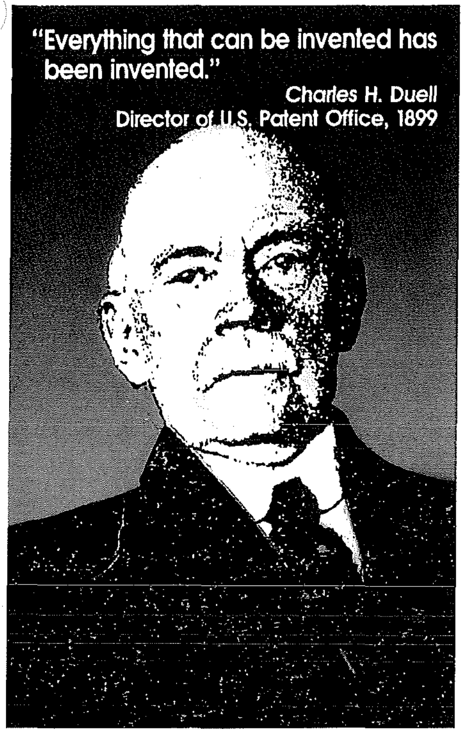

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Director of U.S. Patent Office, 1899

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Harry M. Warner
Warner Bros. Pictures, c. 1927

"Sensible and responsible women do not want to vote."
Grover Cleveland, 1905

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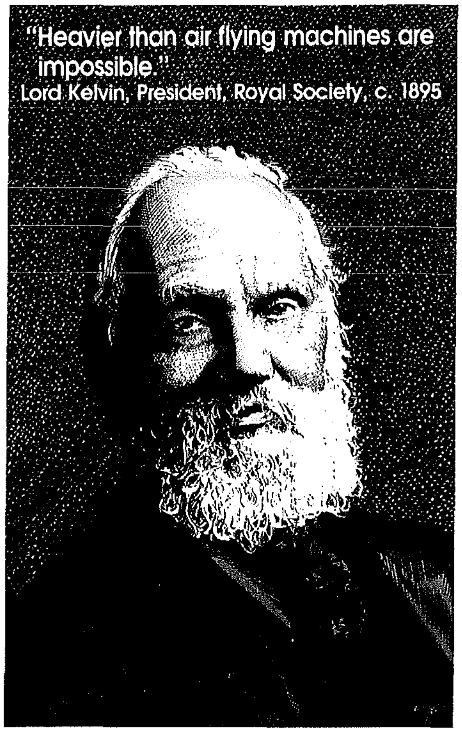
Robert Millikan, Nobel Prize in Physics, 1920

"Ruth made a big mistake when he gave up pitching."
Tris Speaker, 1921



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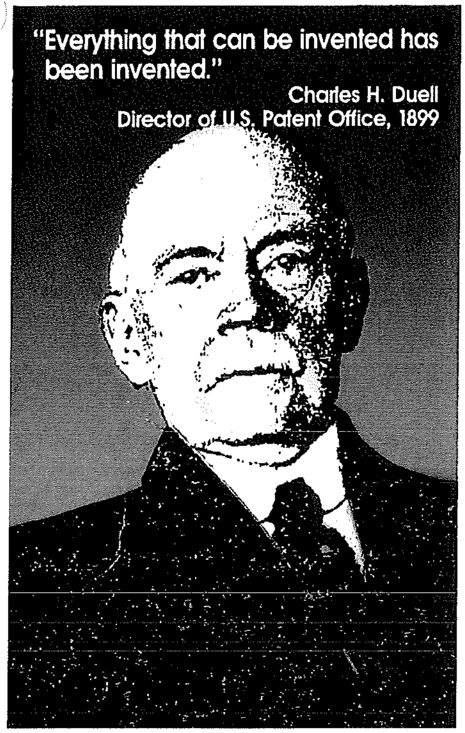
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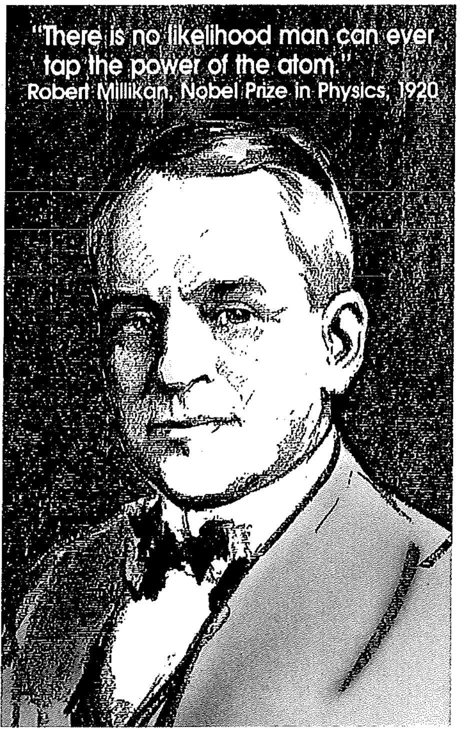
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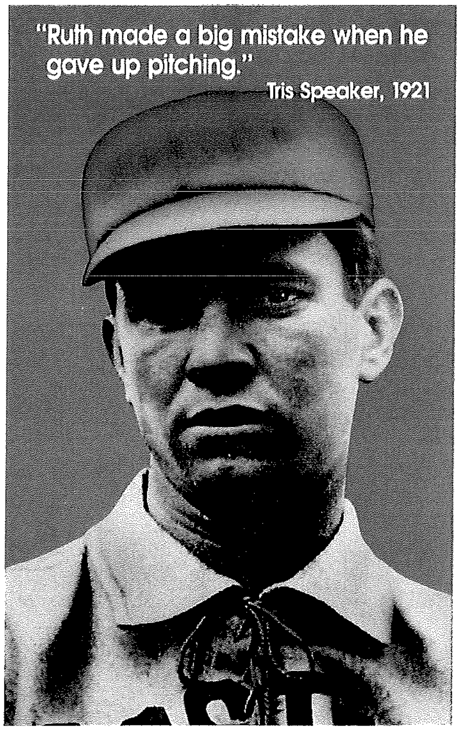
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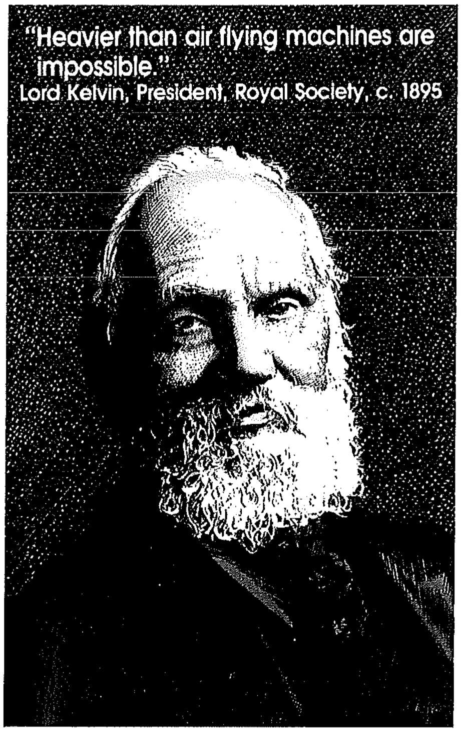
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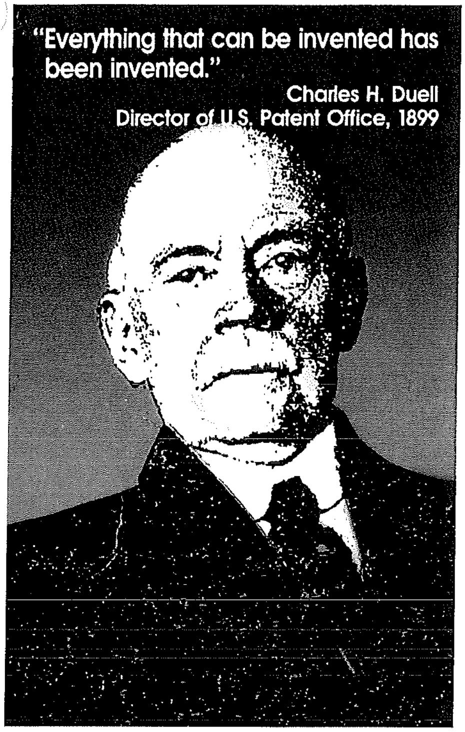

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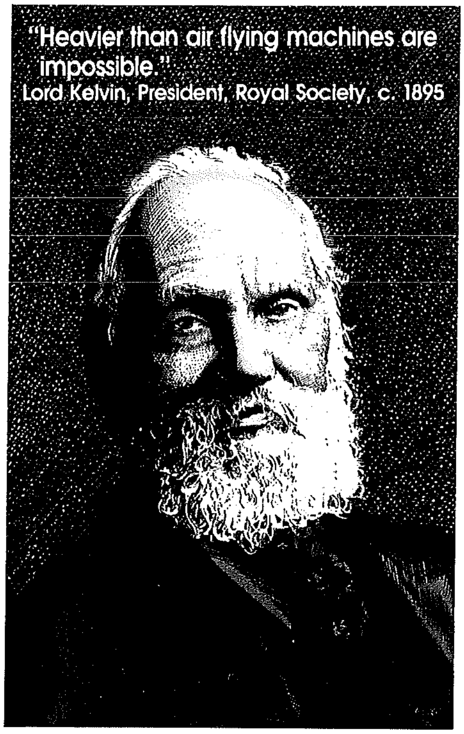

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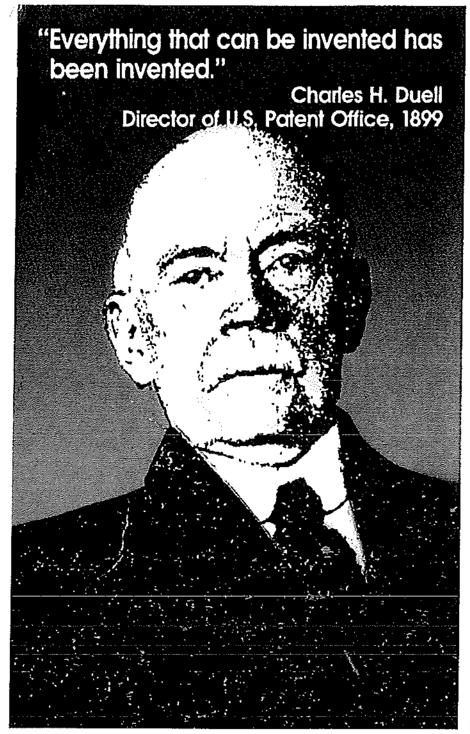
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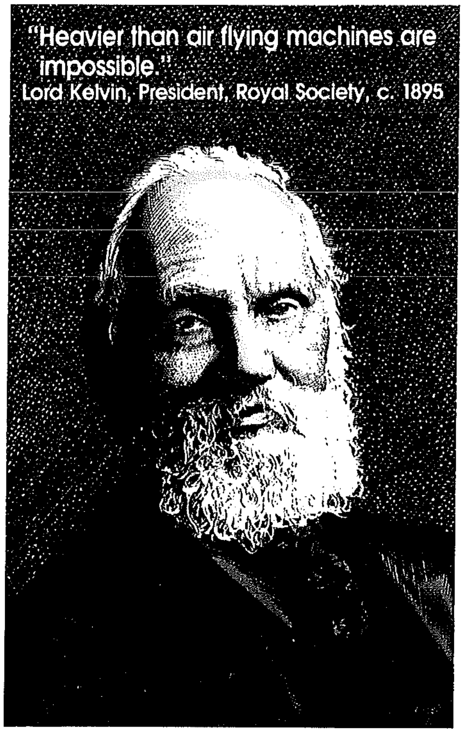
"Heavier than air flying machines are impossible."
Lord Kelvin, President, Royal Society, c. 1895

There's no future
in believing something can't be done.
The future is in making it happen.

"The image which is portrayed (of fuzzy control) is the ability to perform magically well by the new incorporation of 'new age' technologies of fuzzy logic, neural netorks, expert systems, approoximate reasonong, and self organization in the dismal failure of traditional techniques. This is pure, unsupported claptrap which is pretentious and idolatrous in the extreme, and has no place in the scientific literature"

Robert Bitmead
Australian National University
IEEE Control Systems
June 1993, p. 7

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Introduction to Fuzzy Systems

Prof. Bob Marks

★ Hand out Sylabus
★ References
  - Bezdek
  - Klir
  - FUZZ-IEEE

★ Grading
  2 Tests
  Project

★ Course Rules
- No fuzzy jokes
  - Wizard of Fuzz
  - Profuzzer Marks
  - Fuzztival
- No audits
  - Watch on T.V.
  - To hot!
  - Auditing does not work.

ob

Paradigm Shift
Zadeh's Sufferage,
Resist: 'Not invented here'
Difference: 'Wisdom of age vs. babbling of old fools'

"Sensible and responsible women do not want to vote."
Grover Cleveland, 1905

"Who the hell wants to hear actors talk?"
Harry M. Warner,
Warner Bros. Pictures, c. 1927

"that can be invented has
be[illegible]ed."
Charles H. Duell,
Director of U.S. Patent Office, 1899

"Ruth made a big mistake when he gave up pitching."
Tris Speaker, 1921

"Heavier than air flying machines are impossible."
Lord Kelvin, President, Royal Society, c. 1895

"There is no likelihood man can ever tap the power of the atom."
Robert Millikan, Nobel Prize in Physics, 1923

What diffence will fuzzy make?
- simplifying certain complex systems
- Solving non-linear and/or/ time varying systems.
- Controls, Signal Processing

What will stick? (Info Theory, NN's)

"The image which is portrayed (of fuzzy control) is the ability to perform magically well by the new incorporation of 'new age' technologies of fuzzy logic, neural netorks, expert systems, approoximate reasonong, and self organization in the dismal failure of traditional techniques. This is pure, unsupported claptrap which is pretentious and idolatrous in the extreme, and has no place in the scientific literature"

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Bob Bitmead, "On fuzzy control ... and fuzzy reviewing", ***IEEE Control Systems***, vol 13, no.3, pp.5-7 (June 1993).

"Is the assessment the reasoned wisdom of age or the babbling of old fools?"

R.J. Marks II

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"Is the assessment the reasoned wisdom of age or the babbling of old fools?"

R.J. Marks II

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Fuzzy Syst m S
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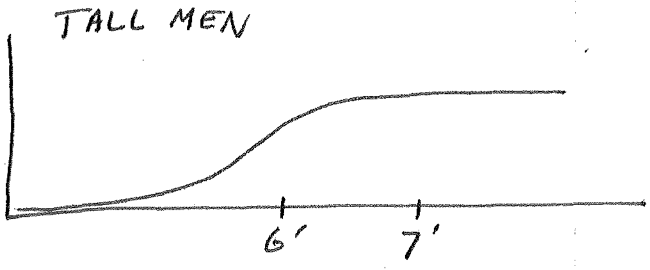
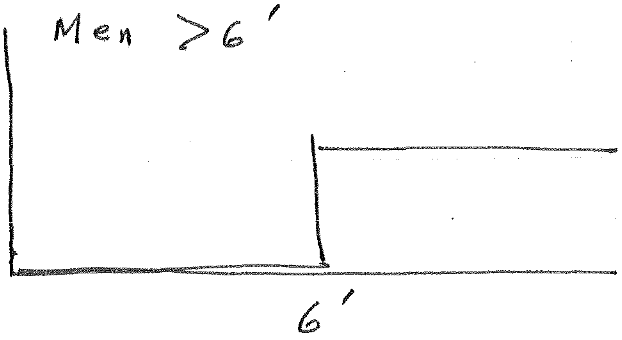
Fuzzy sets:

Grade of membership.



Conventional sets are 'crisp'



Q: What are fuzzy systems

A: A different way to think about things.

A new 'paradigm'

→ Paradigm 'shifts'

★ A Precise Model of Uncertainty ←

Founder: Lotfi Zadeh (1965)

Q: Again: What are fuzzy systems?

A: (Blind man & elephant analogy)

- Linguistically motivated paradymes.

  Example from control

  - Back up 'a lot' or 'a little'

    not

  - Back up 5'6"

  ★ NOTE: Your fuzzy memberships & mine are not same. It still works.

- Management of Complexity

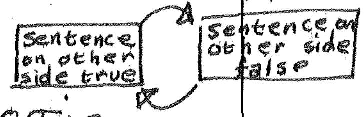
  Car example

  - Details of operation abstracted to fuzzy control. Degree of fuzziness depends on enviornment.
  - Time Varying & Nonlinear Processes

(Kosko)

- Resolution of Bivalent Paradoxes
  - 'Don't believe me!' →
  - 'Nothing's impossible!'

"I shave all men who don't shave themselves" (Who shaves me?) →



$\exists$ no resolution in crisp logic

$\exists$ a resolution if fuzzy logic

(Membership $\equiv \frac{1}{2}$)

Is fuzzy a better model of logical reality?

- The theory of 'possibility'

Q: Isn't fuzziness a clever disguise for probability

A: No!

Probability $\not\equiv$ Possibility

'All thing possible are not probable'
'All things probable are possible'

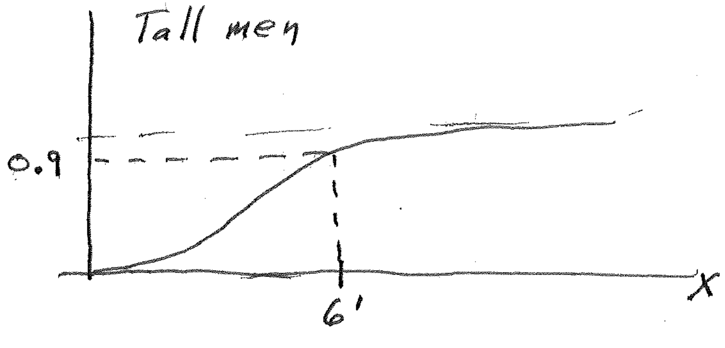
possible $\not\rightarrow$ probable

probable $\rightarrow$ possible

Contrapositive:

impossible $\rightarrow$ improbable

improbable $\not\rightarrow$ impossible



The possibility of a 6' tall man is 0.9

The statement in the box below is

TRUE

The statement in the box above is

FALSE

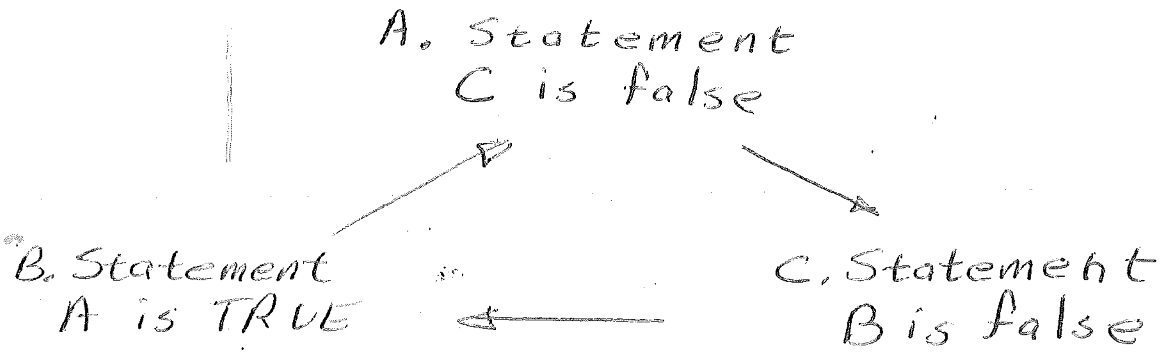
'I NEVER TELL THE TRUTH.'

Townsman Gleason shaves only
the townsmen who do not shave
themselves.

Note: Previous Propositions 50/50

Q: What are examples of non-binary propositions that are other than 50/50?

A: Kosko's analogy breaks down (?)

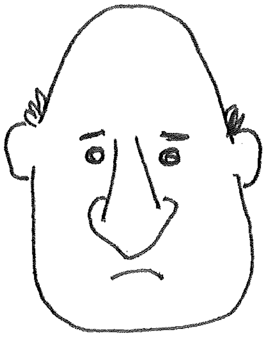


'Trivalent' Paradox.

Degree of truth = ?

Is fuzz = prob? No!

★Example (Rospini)

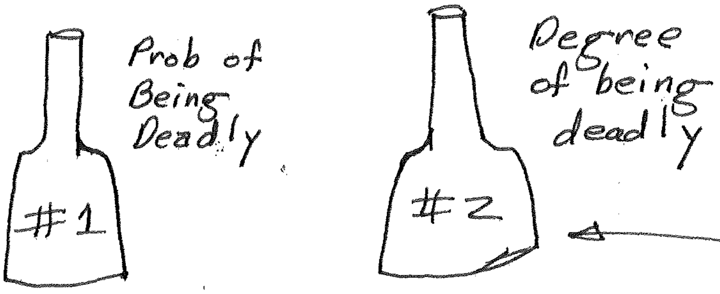


⟶ next p ⟶

possibility of being bald
= membership value in set of
bald men
≠ probability of being bald.

★Example (Bezdek)

You are dying of thirst. Two bottles:



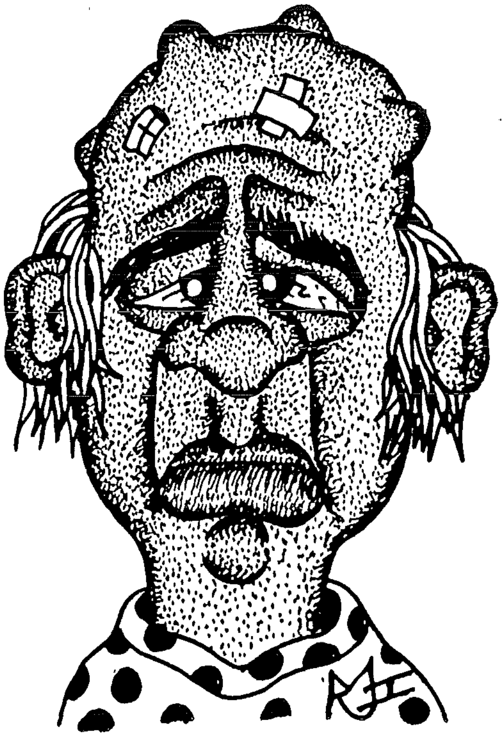
Prob contains deadly poison = 0.1

Degree of content
possibility deadly
poison = 0.1

You would drink bottle 1! (?)

Q: Aren't there techiques other than fuzzy logic to solve any given problem?

A: Yes. One can also solve sinusoidal state circuits w/o using phasors



Because it was smooth, Ed's head frequently slipped off the pillow at night

(The degree of membership of Ed in the fuzzy set of bald men ≠ Probability that Ed is bald).

> "ZADEH'S ORIGINAL PAPER ON FUZZY SETS ... HAS PROBABLY BEEN REPRODUCED THOUSANDS OF TIMES BY NOW AND HAS APPEARED ON VARIOUS OCCASIONS IN SEVERAL OTHER COLLECTIONS OF REPRINTS. THE PAPER IS CLEAR, CONCISE, AND, LIKE ALL REALLY GREAT PAPERS, CONTAINS A WEALTH OF IDEAS THAT HAVE LEAD TO THE ESTABLISHMENT OF NEW BRANCHES OF SCIENCE. THE MOST ASTONISHING THING ABOUT THE PAPER IS THAT ONE CAN GO BACK TO IT, REREAD IT FROM TIME TO TIME, AND FIND GOOD IDEAS FOR CURRENT RESEARCH THAT TO THIS DAY HAVE NOT BEEN FULLY EXPLOITED!"

J.C. Bezdek and S.K. Pal, ***Fuzzy Models for Pattern Recognition***, (IEEE Press, New York, 1992).

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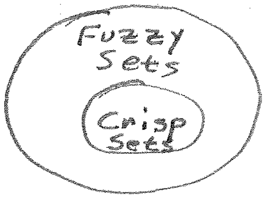
'FUZZY SETS' L.A. Zadeh Inf Control, 1965

$\mathbb{X}$ = set $\quad \mathbb{X} \in \{x\}$

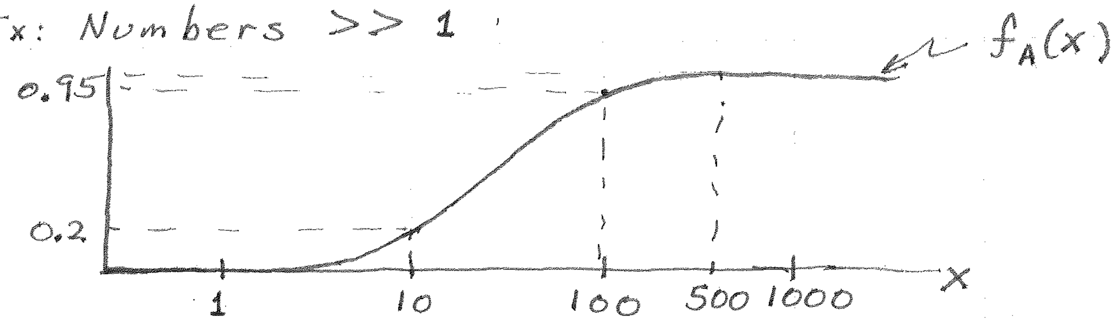
$f_A(x)$ = GRADE OF MEMBERSHIP OF x in A

$0 \le f_A(x) \le 1 \quad \forall \; x$

For a conventional (crisp) set: $f_A = 0$ or 1



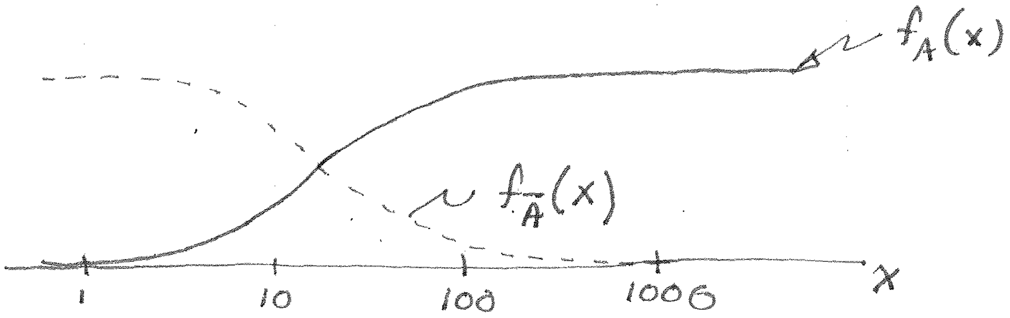
Ex: Numbers >> 1



Definitions: (A, B & C are fuzzy sets)

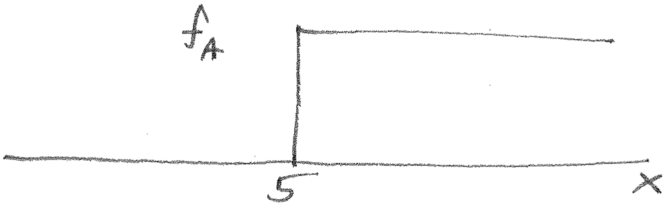
1. A is 'empty' iff $f_A(x) \equiv 0 \quad \forall \; x \in \mathbb{X}$
2. $A = B$ iff $f_A(x) = f_B(x) \quad \forall \; x \in \mathbb{X}$
3. $\bar{A}$ = complement of A

$$f_{\bar{A}}(x) = 1 - f_A(x)$$

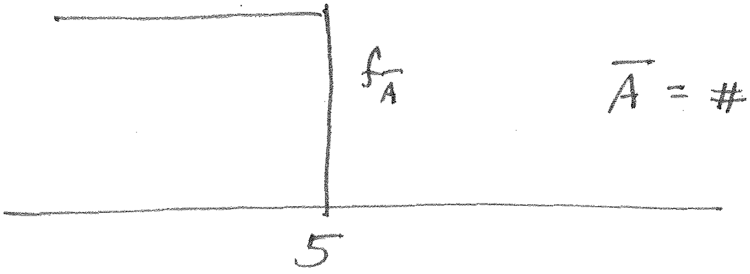


$\bar{A}$ = Numbers not >> 1

Crisp Sets are special cases



$A = \#\text{'s} > 5$



$\bar{A} = \#\text{'s not} > 5$

$$f_{\bar{A}} = 1 - f_A$$

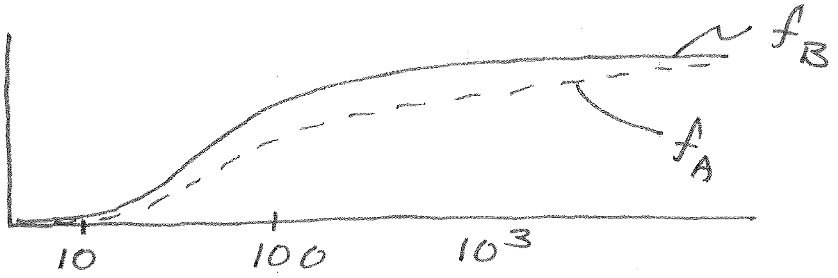
4. Containment

$A \subset B$ (A is a subset of B)

$\Leftrightarrow f_A(x) \leq f_B(x) \quad \forall \; x \in X$

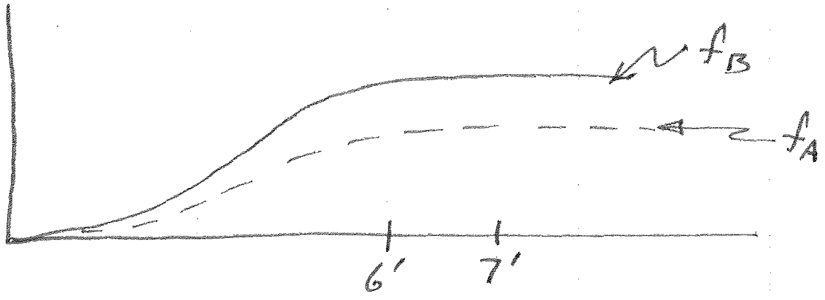
Ex. A = numbers $>> 10$

B = " $>> 1$



Ex: A = very tall men

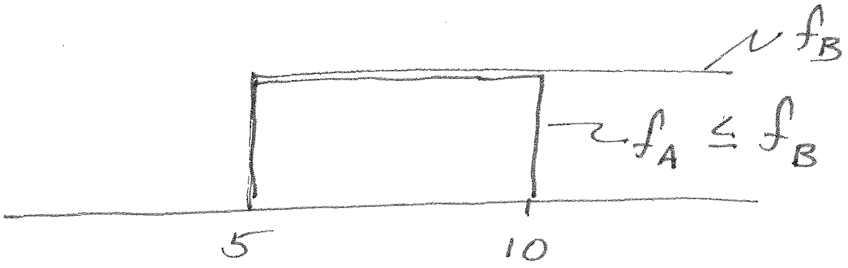
B = tall men



Note: Conventional crisp subsets are special cases

B = #'s > 5

A = #'s > 5 < 10



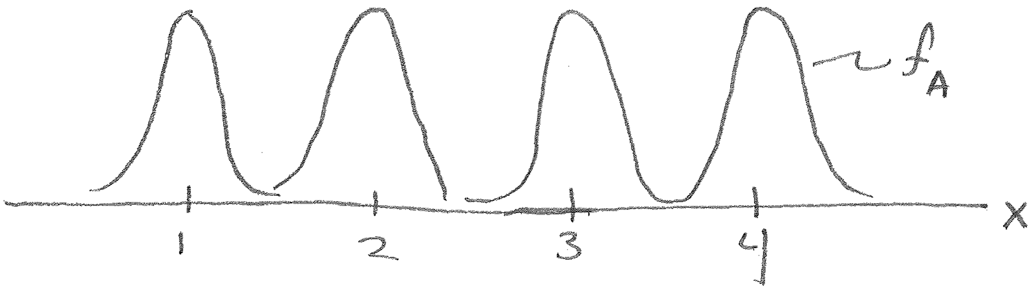
5. Union (Fuzzy 'OR')

$C = A \cup B = A + B$

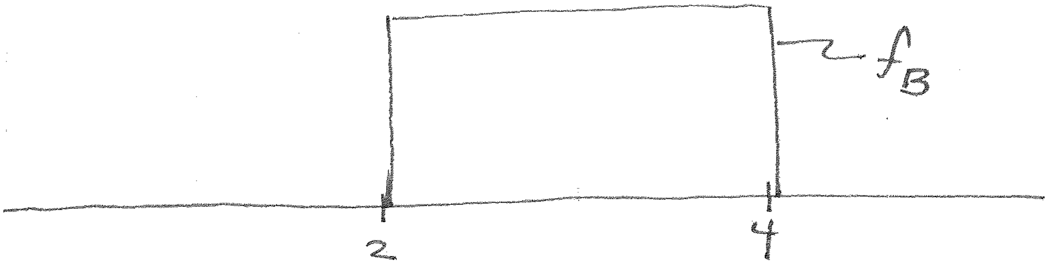
$f_C(x) = \max [f_A(x), f_B(x)]$

$= f_A \vee f_B$

Ex: A = #s close to Integers

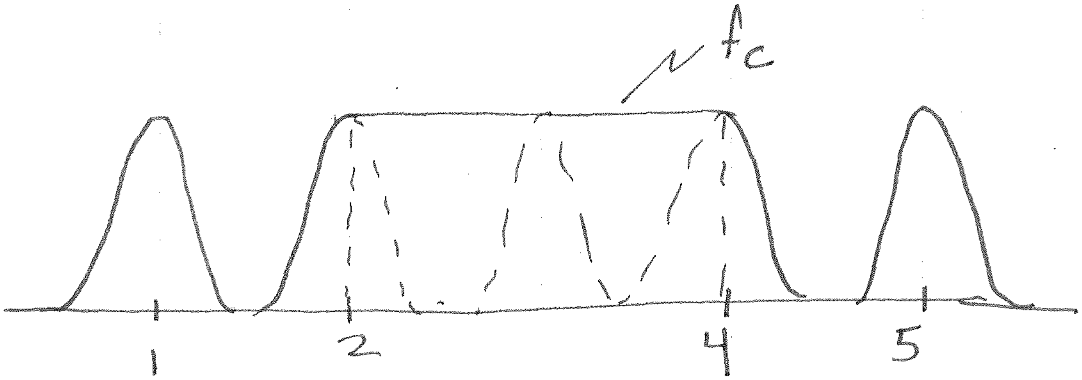


B = #s between 2 & 4 (crisp



C = #'s Close to Integers

<u>or</u> Between 2 & 4



6. Fuzzy Intersection ('And')

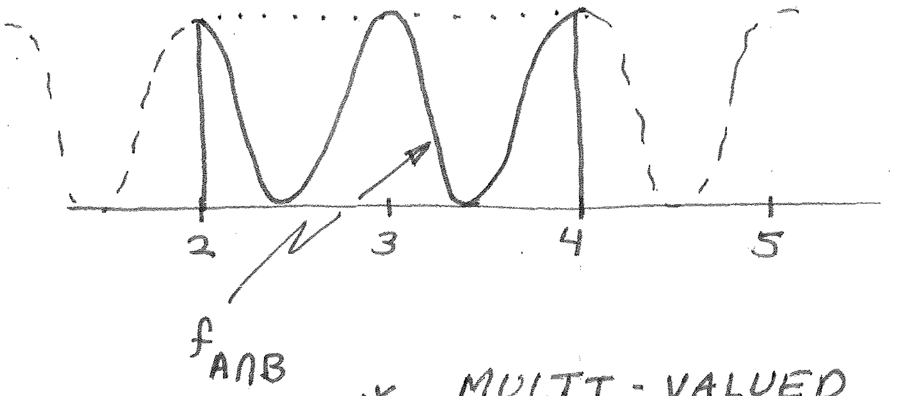
$$C = A \cap B = A \cdot B$$

$$f_C(x) = \min\left[f_A(x),\ f_B(x)\right]$$

Ex

A = #'s close to Integers

B = #'s between 2 & 4



* MULTI-VALUED LOGIC (Next page ⇒)

Notes:

(a) Associative (Elaborate)

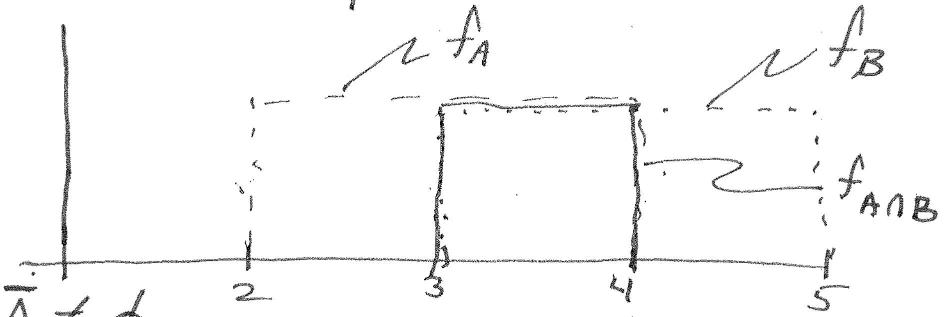
(b) If $A \subset B$, then $B \cap A = \cancel{B}A$ since

$$f_{A\cap B} = \min f_A,\ f_B = f_B$$

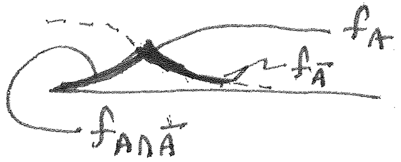
(c) $A\cap B$ is largest fuzzy set contained in both A and B. $A\cap B \subset A$

(d) Crisp special Case

(Use sets on p. 8)



(e) $A \cap \bar{A} \neq \phi$



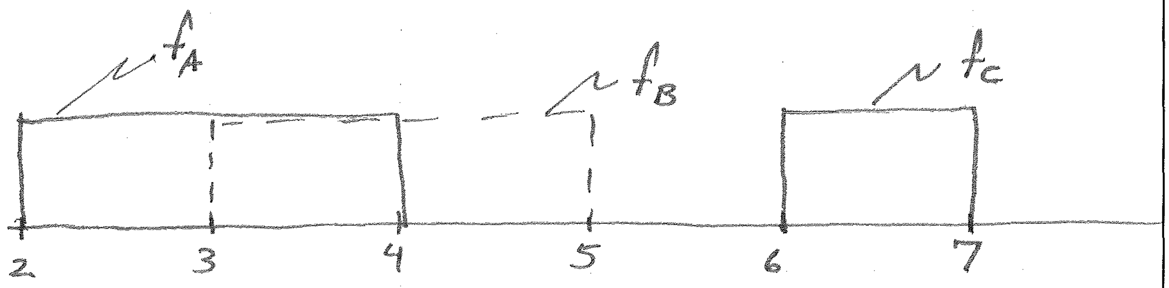
Notes:

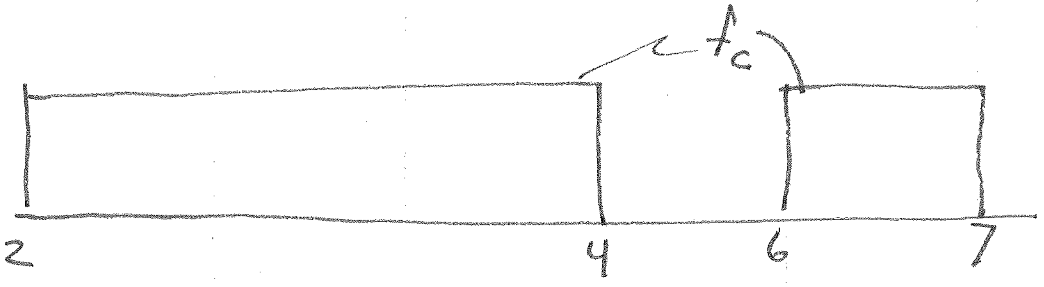
a. Crisp Case

A = #'s between 2 & 4

B = #'s between 3 & 5

C = #'s between 6 & 7





b. Associative

$(A \cap B) \cap C = A \cap (B \cap C)$

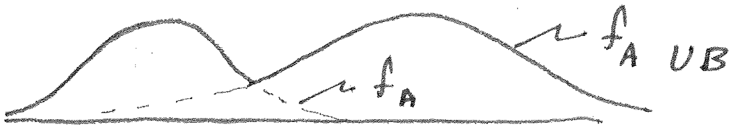
(Follows since MAX is associative)

$\max[(\max(a,b))\,c]$

$= \max[a, \max(b,c)]$

c. $A \subset (A \cup B)$

since $f_A(x) \leq \max[f_A(x), f_B(x)]$



d. $A \cup B$ is the smallest fuzzy set containing both A and B.

Def: C is smaller than D if

$$f_c \leq f_D$$

ie, C is a subset of D

Proof:

$$f_A \leq \max f_A, f_B$$

$$f_B \leq \max f_A, f_B$$

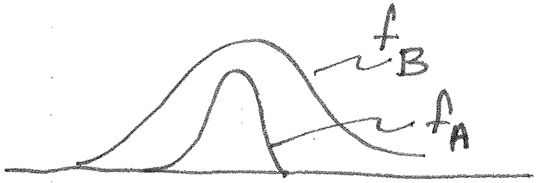
$$f_c = \max f_A, f_B$$

Let $A \subset D$ and $B \subset D$. Then

$$f_D \geq \max f_A f_B = f_c \Rightarrow C \subset D \ \forall D$$

QED

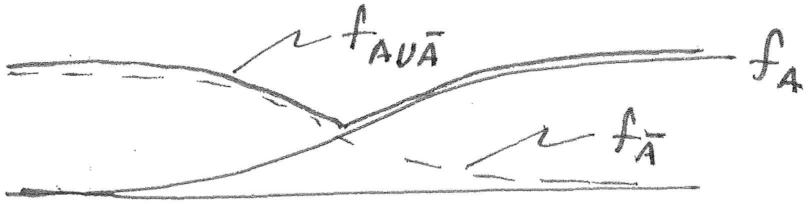
e. If $A \subset B$, then $A \cup B = B$



$$\max f_B, f_A = f_B$$

f. $A \cup \bar{A} \neq$ Universal Set

$$f_{A\cup\bar{A}} = \max [f_A , 1 - f_A]$$



Violates 'Law of excluded middle'
$f_{A\cup\bar{A}} = 1$

$A \cap \bar{A} \neq \phi$
VIOLATES 'LAW OF CONTRADICTION'

Note: $\exists$ other $\cap$ & $\cup$ operations for fuzzy sets.

e.g.

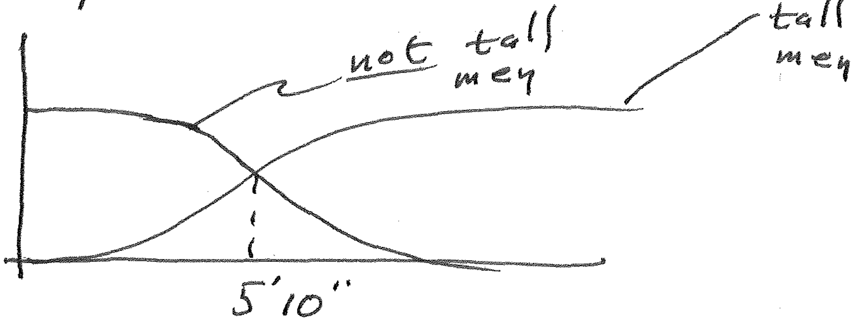
☆ $C = A \cup B \Rightarrow f_C = \min(f_A + f_B, 1)$

note: $A \cup \bar{A}$ = universal set

$f_{A \cup \bar{A}} = 1$

Q: Should this be?

A: Philosophical



Is 5'10" tall? OR

Is 5'10" not tall?

Composite statements truth = 1?

Maybe yes, maybe no

☆ $C = A \cap B \Rightarrow f_C = \sqrt{f_A f_B} \Rightarrow A \cap \bar{A} \neq \phi$

Others in Klir's book

7. Properties of $\cup$, $\cap$ and Complementation

(a) De Morgan's law

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof:

$$f_{\overline{A \cup B}} = 1 - f_{A \cup B}$$

$$= 1 - \max[f_A, f_B]$$

$$f_{\bar{A} \cap \bar{B}} = \min[1 - f_A, 1 - f_B]$$

Q: Are these equal?

A: yes!

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$$f_{\overline{A \cap B}} = 1 - f_{A \cap B}$$

$$= 1 - \min[f_A, f_B]$$

$$f_{\bar{A} \cup \bar{B}} = \max[1 - f_A, 1 - f_B]$$

These are again equal.

(b) Distributive Laws

(i) $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$

$$f_{C\cap(A\cup B)} = \min\left[f_C, \max(f_A, f_B)\right] \quad (1)$$

$$f_{(C\cap A)\cup(C\cap B)} = \max\left[\min(f_A, f_C), \min(f_C, f_B)\right] \quad (2)$$

Are these the same?

$f_A = 1$, $f_B = 2$, $f_C = 3$

(1) is $\min[3, \max(1,2)] = 2$

(2) is $\max[\min(1,3), \min(2,3)] = 2$

Engineer's Proof!

Q: How Prove Rigorously?
A: Exhaustian

$f_A > f_B > f_C$

$f_A > f_C > f_B$

$f_B > f_A > f_C$

$f_B > f_C > f_A$

$f_C > f_A > f_B$

$f_C > f_B > f_A$

(ii) $C \cup (A \cap B) = (C\cup A) \cap C \cup B$

$$f_{C\cup(A\cap B)} = \max\left[f_C \;\; \min(f_A, f_B)\right]$$

$$f_{(C\cup A)\cap(C\cup B)} = \min\left[\max(f_C, f_A), \max(f_C, f_B)\right]$$

Q: True?
A: Yes. Can do exhaustive proof. →

for $f_B > f_c > f_a$

$\max[f_c \min(f_A, f_B)] = \max[f_c, f_B] = f_c$

$\min[\max(f_c f_A), \max(f_c, f_B)]$

$= \min[f_A, f_c] = f_c$ Yes!

8. Algebraic Operations on fuzzy sets

(a) Algebraic Product: $C = AB$

$$f_{AB} = f_A f_B \ ; \ x \in \underline{X}$$

Note:

$$AB \subset A \cap B$$

Why?

$$f_A f_B \leq \min(f_A, f_B)$$

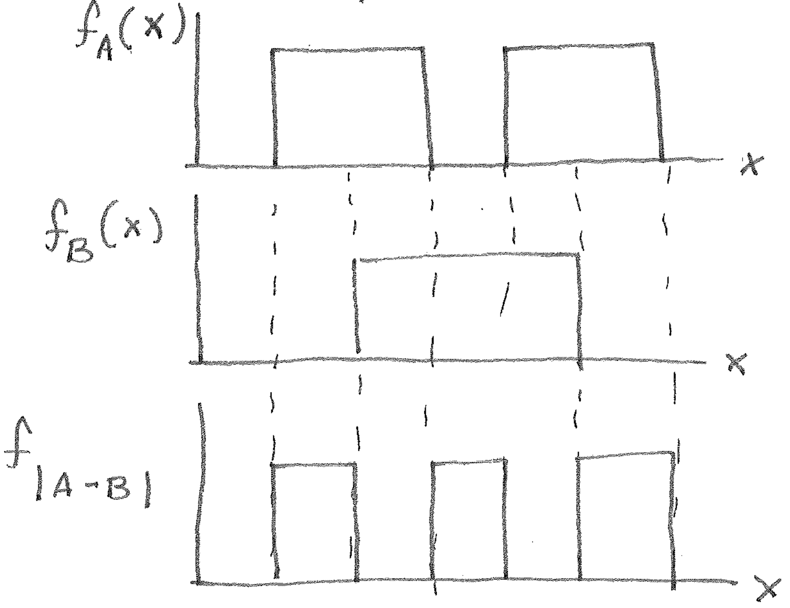
when $0 \leq f_A, f_B \leq 1$

If f_A is min, $f_A f_B < f_A$

(b) Absolute Difference

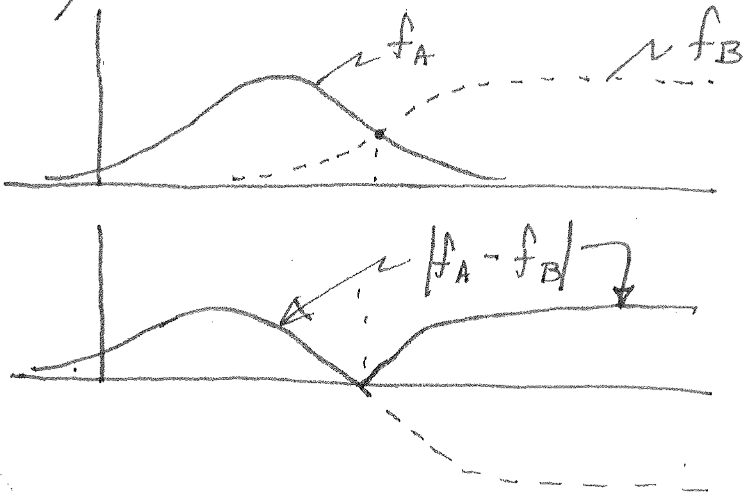
$$C = |A - B| \Rightarrow f_{|A-B|} = |f_A - f_B|$$

* Consider crisp case:

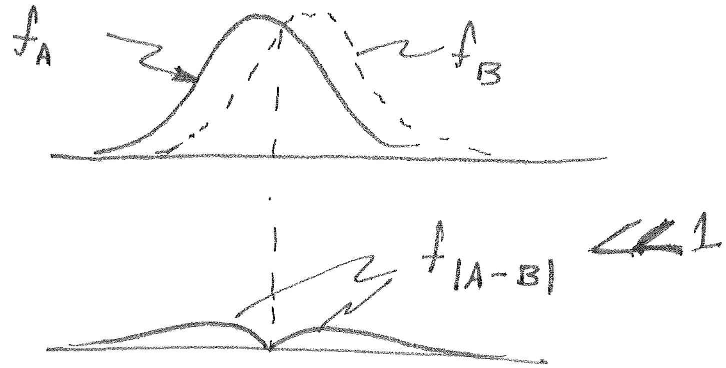


➡ This is an XOR.

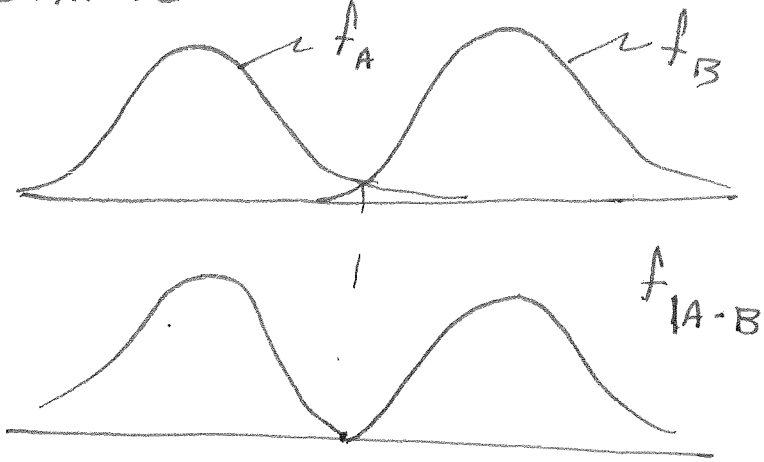
* Fuzzy Case



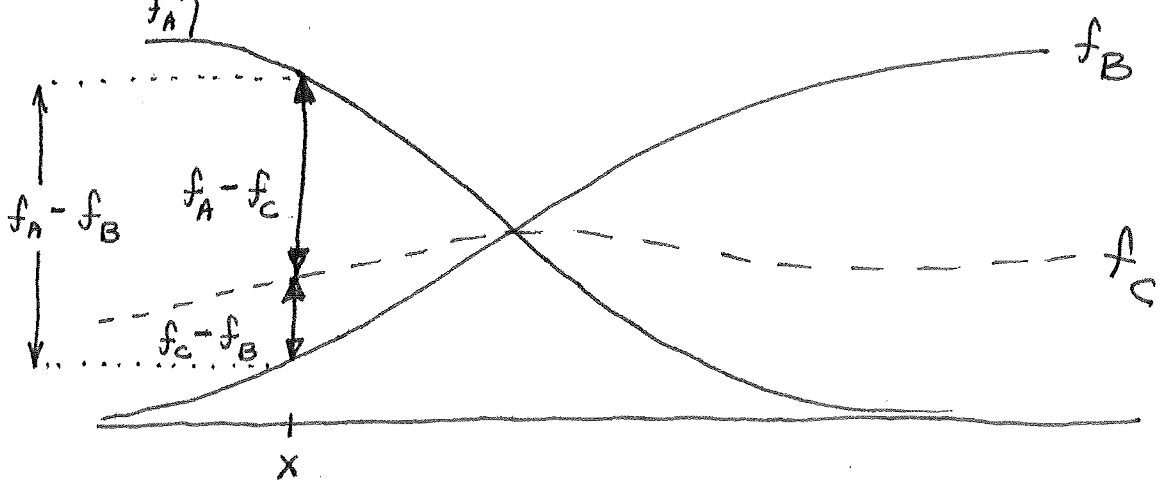
Similar Fuzzy Sets:



Disimilar



$$f_{|A-B|} \approx f_A + f_B$$
$$\approx f_A \cup f_B$$



If $f_A \geq f_B$, then $f_A \geq f_C \geq f_B$

$$f_C = \frac{f_C - f_B}{f_A - f_B} f_A + \frac{f_A - f_C}{f_A - f_B} f_B$$

$$= \frac{f_C - f_B}{f_A - f_B} f_A + \frac{(f_A - f_B) - (f_C - f_B)}{f_A - f_B} f_B$$

$$= \frac{f_C - f_B}{f_A - f_B} f_A + \left(1 - \frac{f_C - f_B}{f_A - f_B}\right) f_B$$

$$= f_\lambda f_A + (1 - f_\lambda) f_B$$

$$\Rightarrow f_\lambda(x) = \frac{f_C(x) - f_B(x)}{f_A(x) - f_B(x)} \quad \text{for } f_A \geq f_B$$

OR Clearly, if $f_A \leq f_B$, $A \leftrightarrow B$ and

$$f_\lambda(x) = \frac{f_C(x) - f_A(x)}{f_B(x) - f_A(x)}$$

Wrong!

Combining

$$f_\lambda(x) = \frac{f_C(x) - f_{A \cap B}(x)}{|f_A(x) - f_B(x)|}$$

(d) Fuzzy relations

'A fuzzy relation in $\mathbf{X}$ is a fuzzy set in the product space $\mathbf{X} \times \mathbf{X}$.'

(Extention to higher dimensions obvious)

(Spaces can be different: ie $\mathbf{X} \times \mathbf{Y}$)

Example (Zadeh, 1972) 'Resemblance' → next p

John  Tom 

Jim  Dick 

} → next p+1

$\downarrow$ $\mathbf{X}$ $\downarrow$ $\mathbf{Y}$

	John	Jim
Tom	0.8	0.6
Dick	0.2	0.9

} relation matrix on $\mathbf{X} \times \mathbf{Y}$

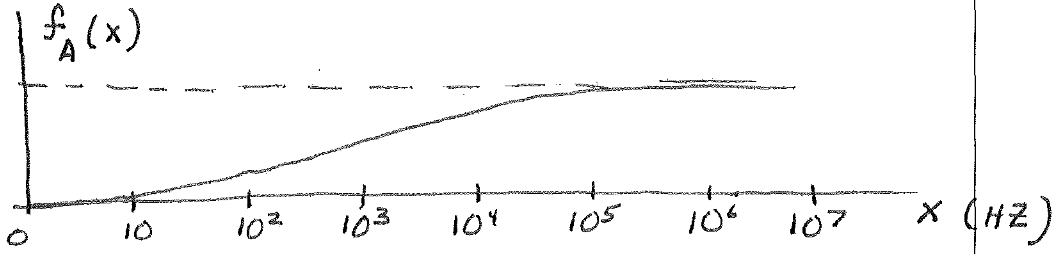
'On a scale from 1 to 10, how much does John resemble Dick?'

(ii) Fuzzy Sets induced by mappings

$$y = T(x) \quad ;$$

$$f_A(x) \xleftrightarrow[y=T(x)]{} f_B(y) \Rightarrow f_A(x) = f_B(y)$$

Example: A = Sinusoids with high frequency



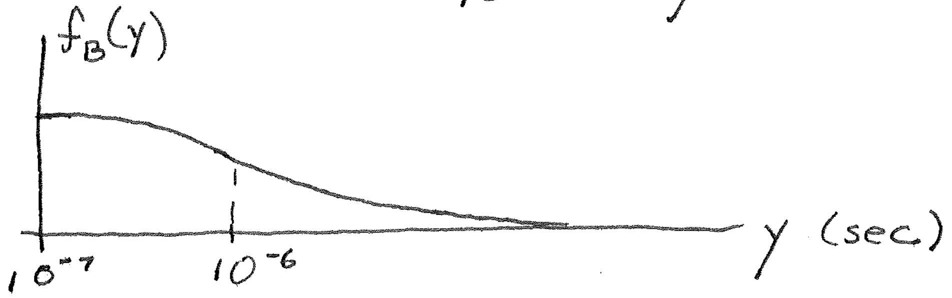
$$f_A(x) = \tanh\left(\frac{x}{10^6 \text{HZ}}\right)$$

B = sinusoids with short periods

$$y = \frac{1}{x} \Rightarrow x = 1/Y$$

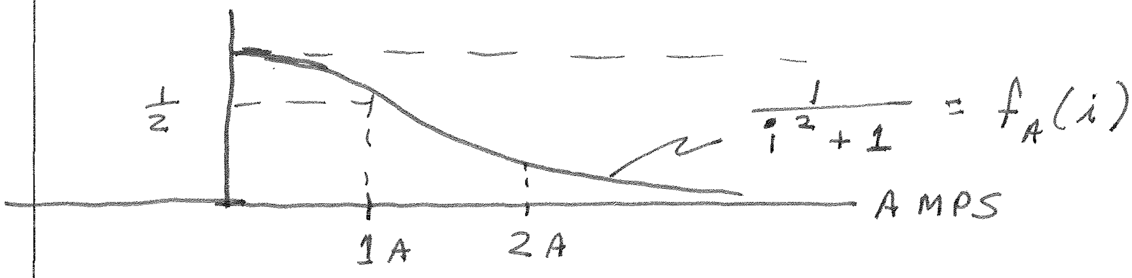
$$f_B(y) = f_A(x) = f_A\left(\frac{1}{Y}\right)$$

$$= \tanh\left(\frac{1}{10^6 \text{HZ } Y}\right)$$



This trasformation is <u>monotonic</u>

Low Positive
~~HIGH~~ Current



2 Ω Resistor ⇒

LOW POWER = $2i^2 = p$

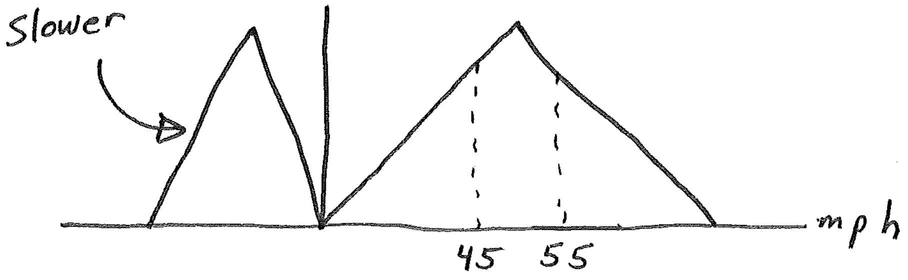
$$f_B(p) = f_A(i) = f_A(\sqrt{p/2})$$

$$f_B(p) = \frac{1}{\frac{p}{2}+1}$$

(still monotonic)
for $i, p > 0$

Nonuique inverse:

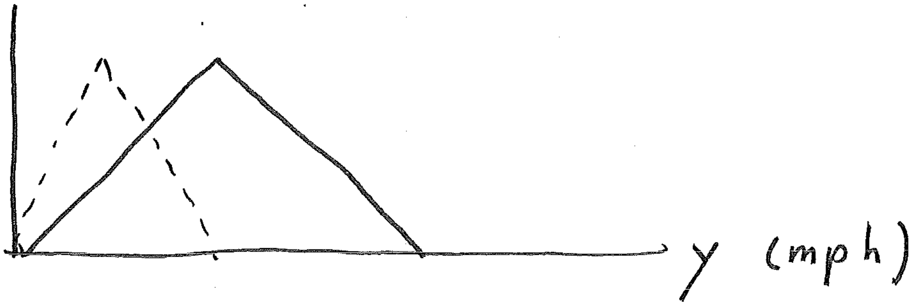
$\underline{X} = \{ x \mid x =$ safe ~~spe~~velocities during morning rush hour$\}$



$y = |x| =$ speed

$\Rightarrow x = \pm y \Leftarrow$ no unique inverse

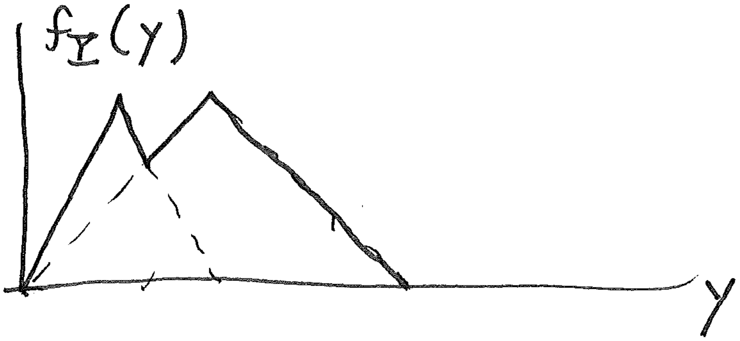
$f_{\underline{Y}}(y) = f_{\underline{X}}(x)$



Q: Which one do we use?

A: From zadeh: max possibility

$\Rightarrow$ use max

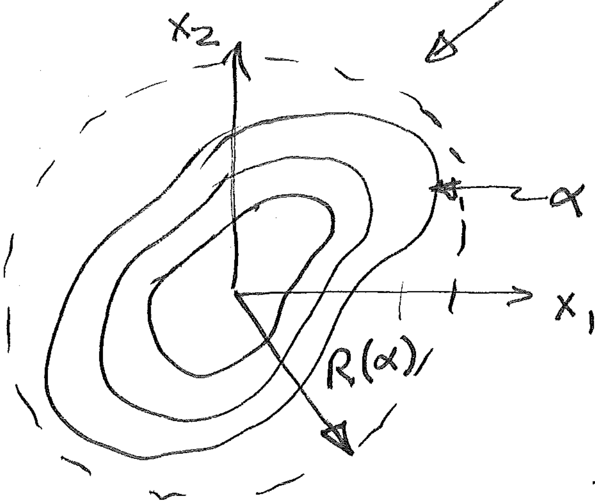


(f) A is bounded if all α cuts are of finite extent, $\cancel{\ni}$

i.e. $\forall \alpha > 0 \ \exists \ R(\alpha) < \infty \ni \|x\| \leq R(\alpha)$ $\forall \ x \in \Gamma_\alpha$

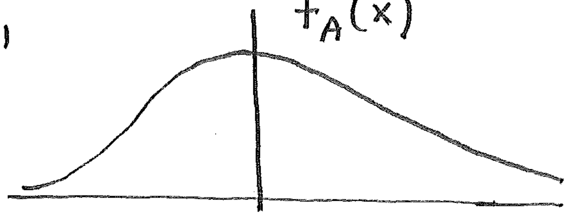
$(\Gamma_\alpha = \{x \mid f_A(x) \geq \alpha\})$

Contour plot



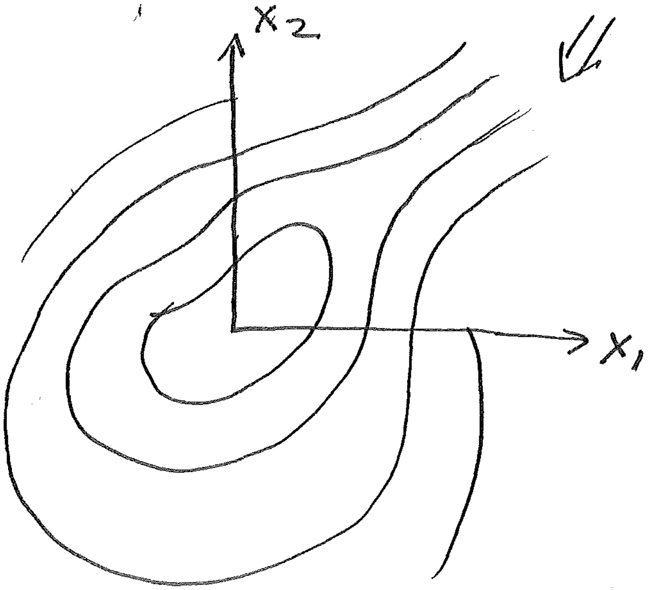
⇐ Bounded

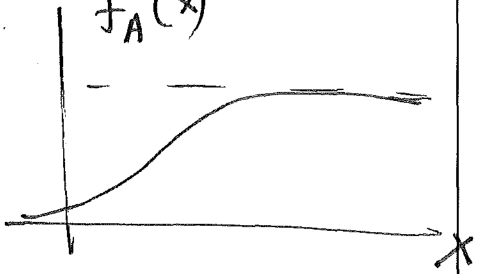
⇓



Not Bounded

⇙ ⇓





A bounded,

B bounded

Then

$A \cap B$ bounded

and $A \cup B$ bounded.

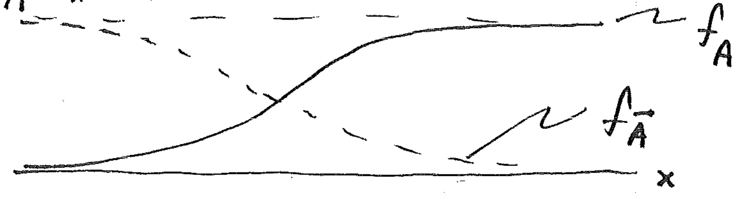
If A is not bounded

and B is not "

? $\Rightarrow A \cap B \rightarrow$?

$A \cup B$ is bounded Λ

Ex: $A = \#$'s $>> 1$



A & $\bar{A}$ are not bounded

but $A \cap \bar{A}$ is

A={Lakes that are deep and wide ← At widest point}

deep

		500'	1000'	5000'	>>5000'
wide	1/2 mile	0.1	.2	.3	.4
	5 miles	.2	.3	.4	.5
	50 miles	.3	.5	.7	.8
	>>50 miles	0.4	.6	.8	1

⟸ f_A

Can we find $f_B \Rightarrow B =$ deep lakes?
Yes! Take fuzzy 'marginals' (projection) or shadow

deep

500'	1000'	5000'	>>5000'
0.4	0.6	0.8	1

or

$$f_A(x) = 0.4/500' + 0.6/1000' + 0.8/5000' + 1/>>5000'$$

Lectures From Klir & Folger

Assignment: Read Chapter 1

Chapt 2: Operations on Fuzzy Sets

2.1 General Discussion

Standard Operations of Fuzzy Set Theory

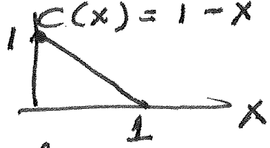
$$\mu_{\bar{A}} = 1 - \mu_A$$
$$\mu_{A \cup B} = \max[\mu_A, \mu_B]$$
$$\mu_{A \cap B} = \min[\mu_A, \mu_B]$$

There exist other operations in fuzzy set theory. Their required properties can be stated axiomatically.

Discussion:

Shannon gave eloquent argument why log (PROB) measures information. He justified why log was only function. We will also argue for min & max from axiomatic arguments!

Standard Compl. ⇒



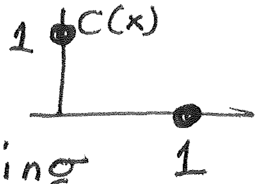
2.2. Fuzzy Complement

$c: [0,1] \longrightarrow [0,1]$

$c[\mu_A(x)] =$ membership of complement of A

Axiom c1: $c(0) = 1$, $c(1) = 0$

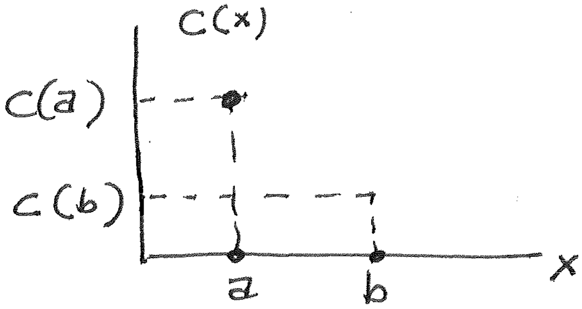
Boundary Conditions



Axiom c2: Monotonic nonincreasing

$\forall\ (a,b) \in [0,1]$, ~~∈~~

$a < b \Rightarrow c(a) \geq c(b)$



These are 'Axiomatic Skeleton' for fuzzy complement. Sufficient for crisp set.

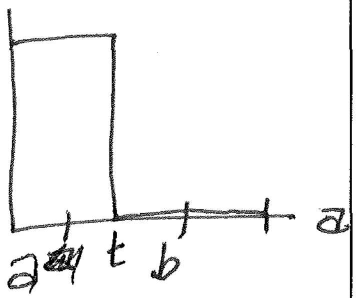
Example (Bad)

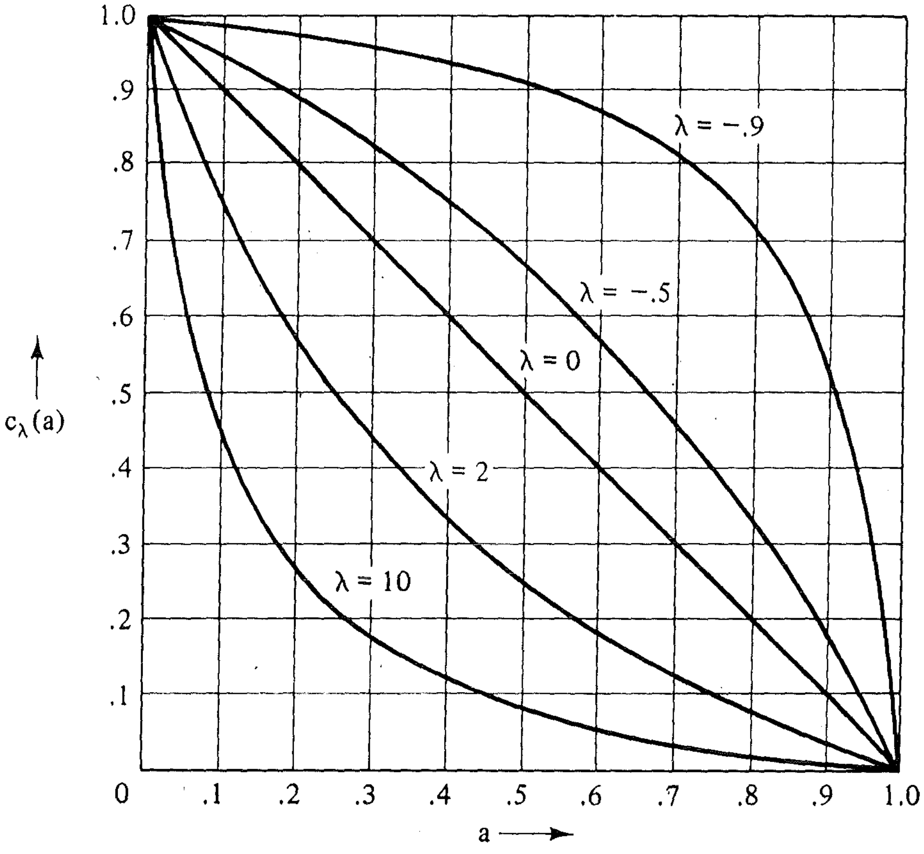
$$c(a) = \begin{cases} 1 \ ; & a \leq t \\ 0 \ ; & a > t \end{cases}$$

Proof: $c(0) = 1$, $c(1) = 0$

$c(a) \geq c(b)$

when $a \leq b$





(a)

SUGENO CLASS

Additional Axioms:

Axiom C3: C is continuous

Axiom C4: C is involutive. ie

$$c\left(c(a)\right) = a \quad \forall \quad 0 \le a \le 1 \}$$

* SUGENO CLASS: $\lambda \in (-1, \infty) \doteq$ Parameter

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a}$$

{SHOW FIG 2.3a on p.42

'STANDARD' IS $\lambda = 0$

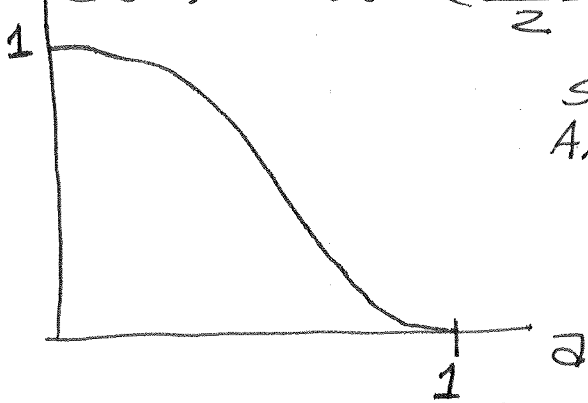
$$c_\lambda\left(c_\lambda(a)\right) = \frac{1 - \frac{1-a}{1-\lambda a}}{1 + \lambda \frac{1-a}{1-\lambda a}}$$

$$= \frac{(1-\lambda a)-(1-a)}{(1-\lambda a) + \lambda - \lambda a}$$

$$= \frac{(1-\lambda)a}{(1-\lambda)} = a$$

* One that doesn't work:

$$c(a) = \cos^2\left(\frac{\pi a}{2}\right)$$



satisfies Axioms 1 & 2 & 3

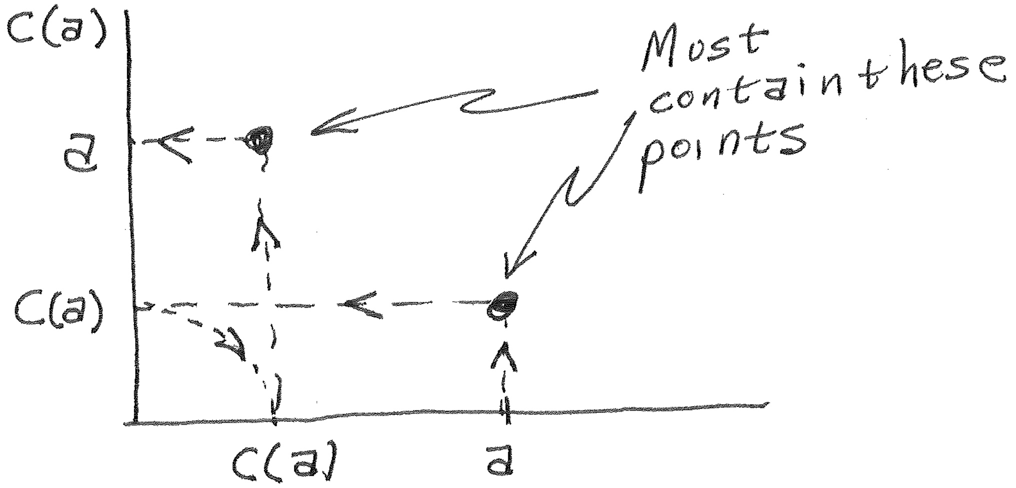
BUT

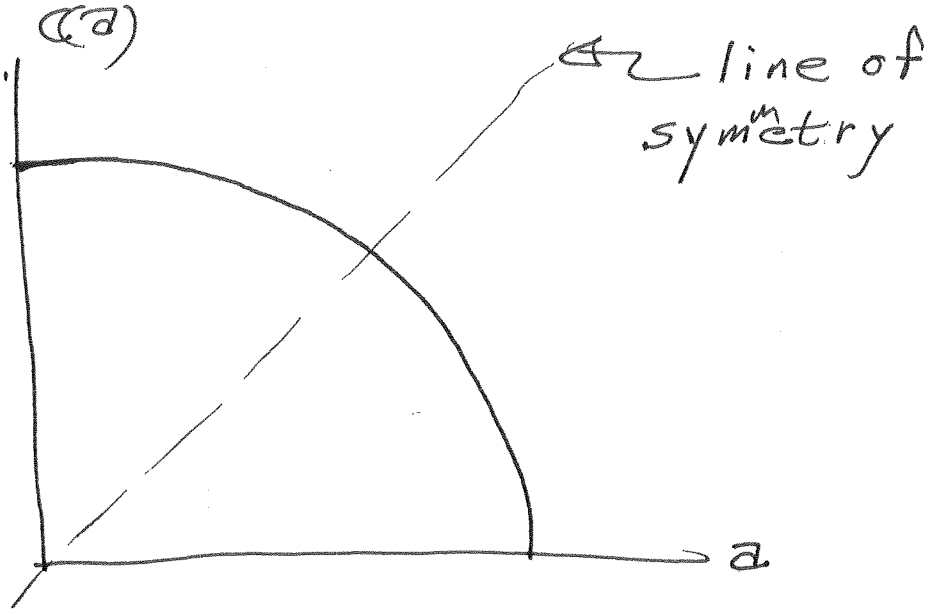
$$c\left[c(a)\right] = \cos^2\left[\frac{\pi}{2}\cos^2\left(\frac{\pi a}{2}\right)\right] \ne a$$

not involutive

Required Symmetry:

$a = c(c(a))$



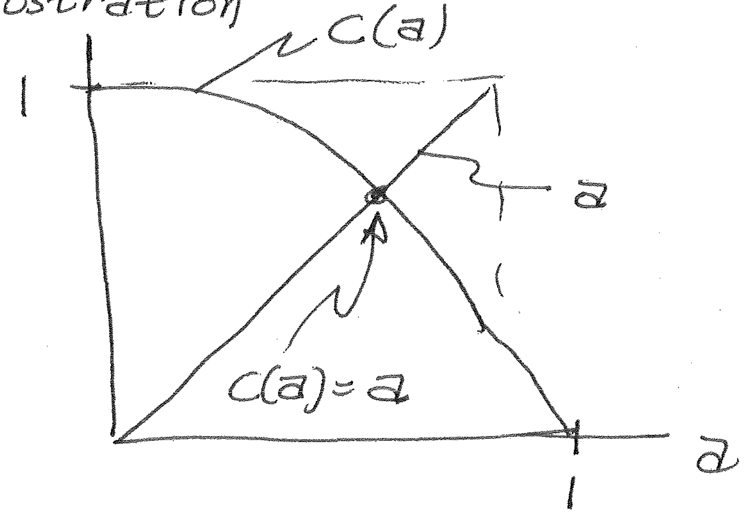


Defn: Equalibrium point

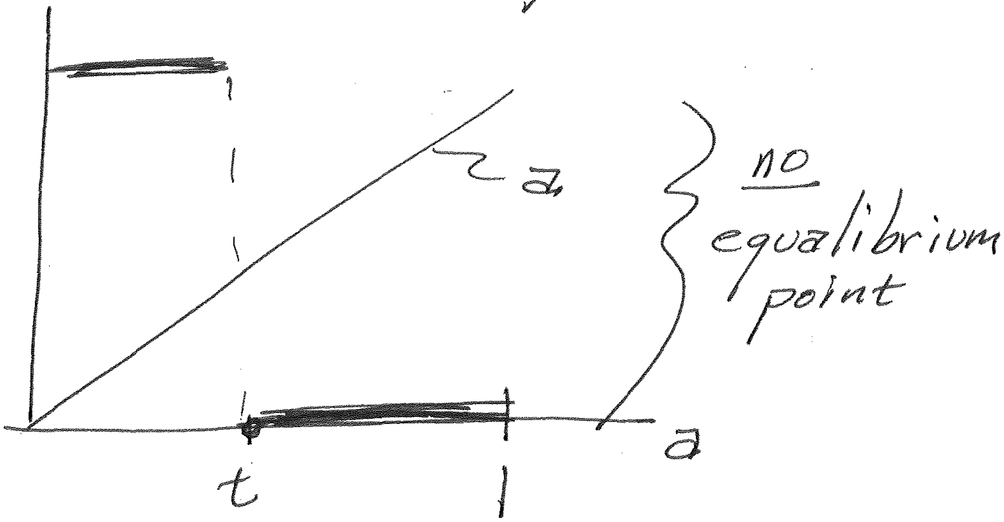
$$c(a) = a$$

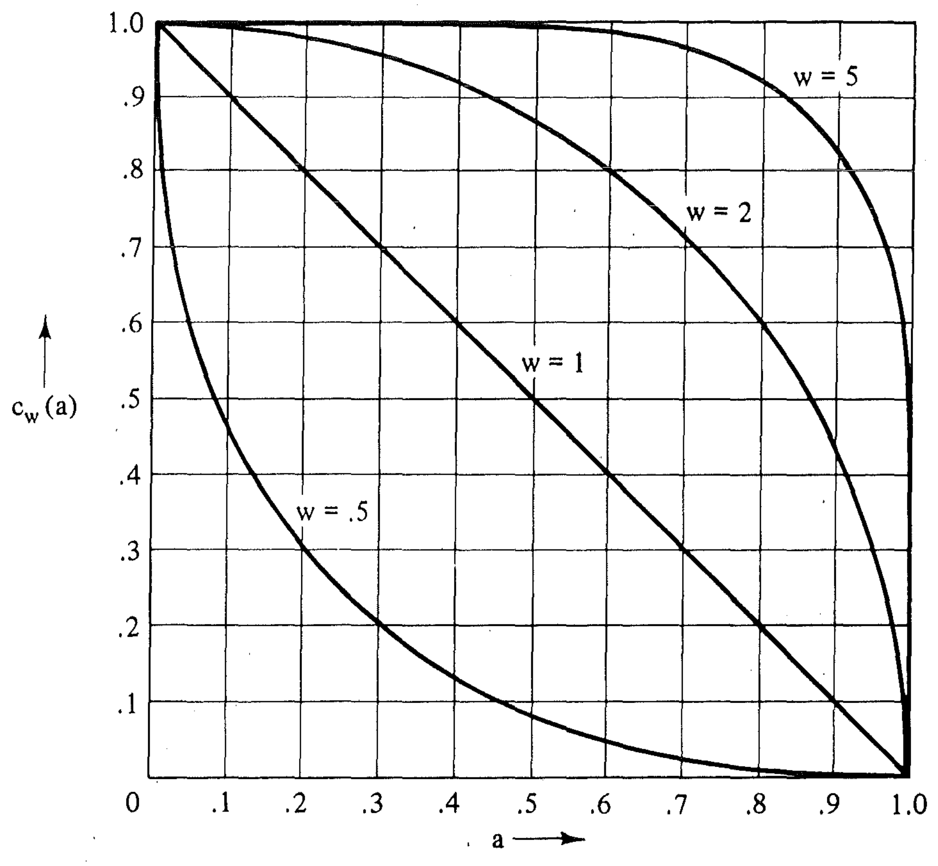
Theorem: Every fuzzy complement has, at most, one equalibrium

Illustration



Only one, since monotonically nondecreasing





(b)

Figure 2.3. Examples from two classes of involutive fuzzy complements: (a) Sugeno class; (b) Yager class.

* YAGER CLASS

'STANDARD' IS W=1

$$C_W(a) = (1 - a^W)^{\frac{1}{W}} \qquad 0 < W < \infty$$

SHOW FIG 2.3b on p.42

$$C_W(C_W(a)) = C_W\left((1 - a^W)^{\frac{1}{W}}\right)$$

$$= \left(1 - \left[(1 - a^W)^{\frac{1}{W}}\right]^W\right)$$

$$= 1 - (1 - a) = a \leftarrow$$

involution property holds!

Them 2.3: If c is a continuous fuzzy complement, then c has a unique equalibium

Proof

Consider

$$b = c(a) - a$$

$$c(0) - 0 = \underline{1}$$

$$c(1) - 1 = -1$$

Since $b(a)$ is continuous,

$$\exists\, a \mid 0 \le a \le 1 \ni \qquad b(a) = 0$$

$$\Rightarrow c(a) = a$$

(intermediate value theorem for continuous functions)

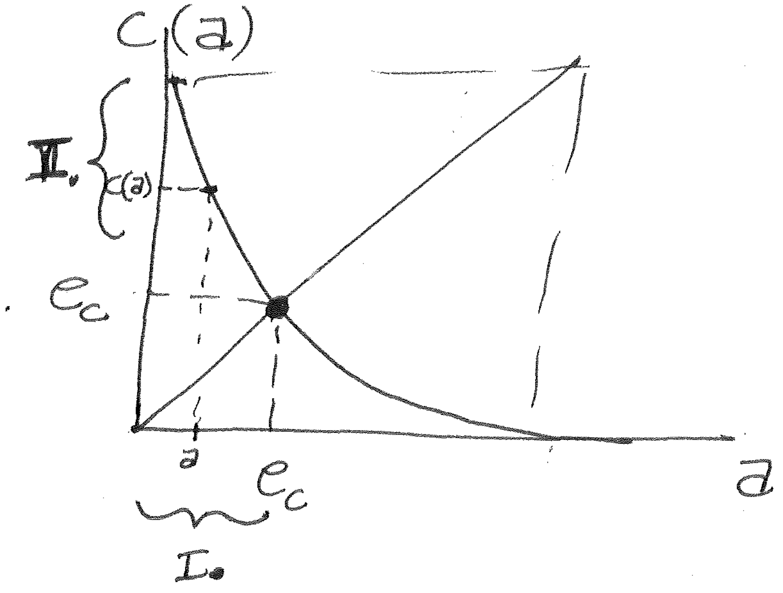
By Them 2.1)

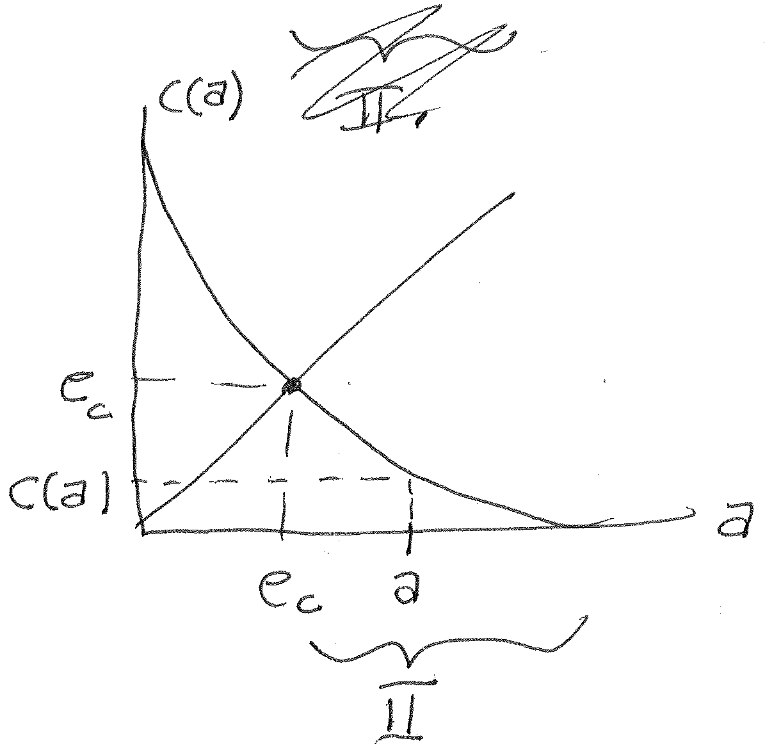
Theorem 2.2

If c has a (unique) equalibrium point, e_c, then

I. $a \leq c(a)$ iff $a \leq e_c$

II. $a \geq c(a)$ iff $a \geq e_c$





Follows from monotinicity

Equalibriom for Sugeno Class

$$c(a) = \frac{1-a}{1+\lambda a}$$

Equalibrium for Sugeno Class:

$$\frac{1-a}{1+\lambda a} = a$$

$$1-a = (1+\lambda a)a$$

$$= a + \lambda a^2$$

$$\Rightarrow \lambda a^2 + 2a - 1 = 0$$

$$a = \frac{-2 \pm \sqrt{4+4\lambda}}{2\lambda} = e_c$$

$$\Rightarrow e_c = \frac{\sqrt{1+\lambda} - 1}{2\lambda} \Leftarrow \text{on p 44 FIG 2.4}$$

Note: For $\lambda = 0$,

$$c(a) = 1 - a = a \Rightarrow e_c = a = \frac{1}{2}$$

41a

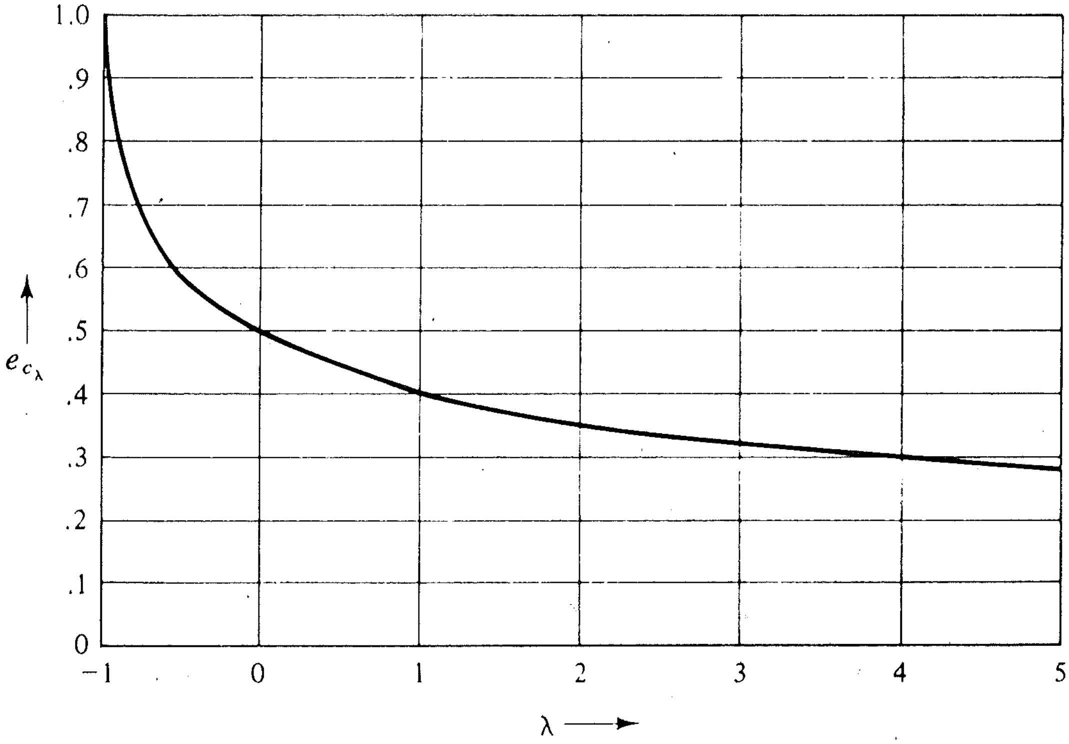


Figure 2.4. Equilibria for the Sugeno class of fuzzy complements.

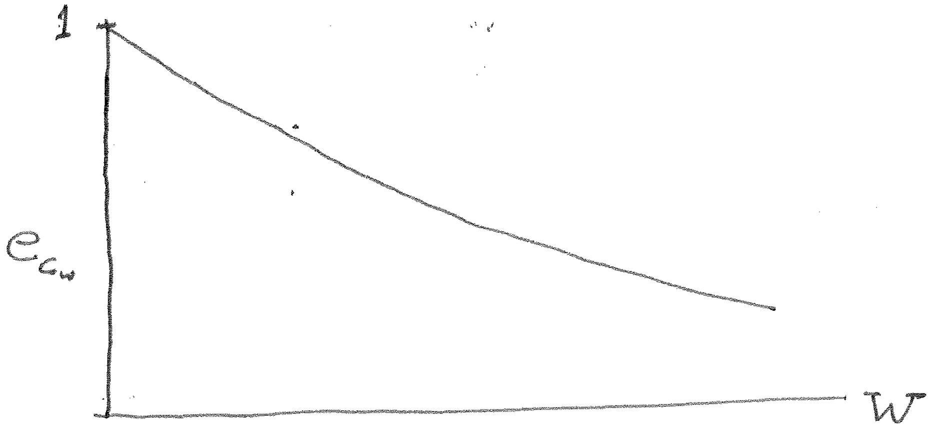
Yager Class:

$$C(a) = (1 - a^w)^{\frac{1}{w}} = a$$

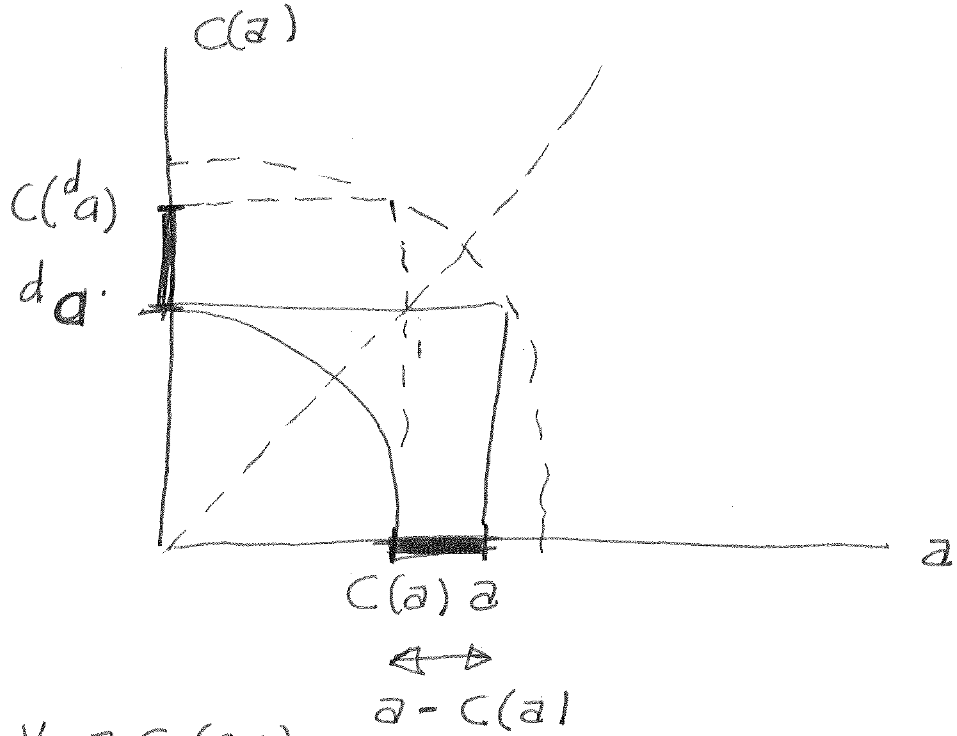
$$1 - a^W = a^W$$

$$a^W = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{1/W}$$



Dual Point:



$\forall\ a \in (0,1)$

$\exists$ a unique point ${}^d a$

(the dual of a)

$$\ni \quad C({}^d a) - {}^d a = a - C(a)$$

If c is involutive, then ${}^d a = c(a)$

2.3. Fuzzy Union

$$u: [0,1] \times [0,1] \longrightarrow [0,1]$$

$$\mu_{A \cup B}(x) = u[\mu_A(x), \mu_B(x)]$$

'STANDARD' UNION → $\mu_{A \cup B} = \max[\mu_A, \mu_B]$

AXIOMATIC SKELETON

Axiom U1: Crisp Boundary Conditions

$$u(0,0) = 0$$

$$u(0,1) = u(1,0) = u(1,1) = 1$$

Axiom U2: COMMUTATIVE

$$u(a,b) = u(b,a)$$

Axiom U3: Monotonic

$$\underbrace{a \le a', b \le b'}_{\text{Grade of membership}} \Rightarrow u(a,b) \le u(a',b')$$

Axiom U4: Associative

$$u(u(a,b),c) = u(a,u(b,c))$$

Additional Requirements:

Axiom U5: u is continuous.

Axiom U6: Indempotence

$$u(a,a) = a$$

Example of first five axioms:

YAGER class:

$$u_w(a,b) = \min\left[1, (a^w + b^w)^{\frac{1}{w}}\right]$$

$$; 0 < w < \infty$$

$\Rightarrow$ not indempotent

$$u_1 = \min[1, a+b]$$

$$u_2 = \min\left[1, \sqrt{a^2 + b^2}\right]$$

$$\vdots$$

$$u_\infty = \max(a,b) \Leftarrow \text{STANDARD UNION}$$

What! $\Rightarrow$

ℓ_N norms:

$$\|\vec{x}\|_N = \sqrt[N]{\sum_n x_n^N}$$

$$\|\vec{x}\|_\infty = \max_n |x_n|$$

Why? $\vec{x} = \begin{bmatrix} 10 \\ 10^2 \\ 10^3 \end{bmatrix}$

$$\|\vec{x}\|_{10} = \sqrt[10]{10^{10} + 10^{20} + 10^{30}}$$

$$\doteq \sqrt[10]{10^{30}} = 10^3$$

$$= \max_n |x_n|$$

Proof: (Assume W.L.O.G. that $a<b$)

$$\lim_{w\to\infty} \ln (a^w + b^w)^{1/w}$$

$$= \lim_{w\to\infty} \frac{\ln[a^w + b^w]}{w}$$; since $a, b < 1$

$\Rightarrow a^w, b^w << 1$

$\Rightarrow \ln[a^w + b^w] \to -\infty$

Use: l'Hospital

$$\lim_{w\to\infty} \ln (a^w + b^w)^{1/w}$$

$$= \lim_{w\to\infty} \frac{\ln e^{w\ln a} + e^{w \ln b}}{w}$$

$$= \lim_{w\to\infty} \frac{(\ln a) e^{w\ln a} + (\ln b) e^{w \ln b}}{e^{w\ln a} + e^{w\ln b}}$$

$$= \lim_{w\to\infty} \frac{a^w \ln a + b^w \ln b}{a^w + b^w} = \lim_{w\to\infty} \frac{\left(\frac{a}{b}\right)^w \ln(a) + \ln(b)}{\left(\frac{a}{b}\right)^w + 1}$$

Since $a<b$ $= \ln b$ QED

Thus

$$u(a,b) = \min\left[1, (a^w + b^w)^{\frac{1}{w}}\right]$$

$$\xrightarrow[w \to \infty]{} \min\left[1, \max(a,b)\right]$$

$$= \max(a,b) \quad \leftarrow \text{Theorem 2.6}$$

There are other unions (than Yager's)

(see p. 50)

2.4. FUZZY INTERSECTION

$$i : [0,1] \times [0,1] \longrightarrow [0,1]$$

$$\mu_{A \cap B}(x) = i\,[\mu_A(x), \mu_B(x)]$$

Axiomatic skeleton

Axiom i1: Crisp Boundary Conditions

$$i(1,1) = 1; \quad i(0,1) = i(1,0) = i(0,0) = 0$$

Axiom i2: Commutative

$$i(a,b) = i(b,a)$$

Axiom i3: Monotonic

$$a \le a',\ b \le b' \Rightarrow i(a,b) \le i(a',b')$$

This intersection smaller

Axiom i4: Associative

$$i(i(a,b),c) = i(a, i(b,c))$$

Other:

Axiom: i5

i is a continuous function

Axiom i6: Idempotence

$i(a,a) = a$

Yager Class: (all but idempotent)

$$i_w(a,b) = 1 - \min\left[1, \sqrt[w]{(1-a)^w + (1-a)^w}\right]$$

$$(0 < w < \infty)$$

Note:

$$i_\infty(a,b) = 1 - \min\left[1, \max\ (1-a), (1-b)\right]$$

$$= 1 - \max\left[(1-a), (1-b)\right]$$

$$= \min[a,b] \Longleftarrow$$ 'Standard' Intersection

2.5. Combination of Operations

Lemma. From skeletal axioms,

$u(a,0) = a$

Proof:

$u(\alpha, u(0,0)) = u(u(\alpha,0),0)$ ← Assoc

$u(\alpha,0) = \underbrace{u(u(\alpha,0),0)}_{= u(\alpha, u(0,0)) \text{ assoc}}$ ← Boundary

Let $u(\alpha,0) = a$

$\Rightarrow a = u(a,0)$ COTE!

Similarly

$u(a,1) = 1$

Proof

$u(\alpha, u(1,1)) = u(u(\alpha,1),1)$ ← Assoc

$u(\alpha,1) = u(u(\alpha,1),1)$ ← Boundary Cond

$a = u(\alpha,1)$

$\Rightarrow a = u(a,1)$

Also...

$i(0,a) = 0$

$i(1,a) = a$

similar proofs

TABLE 2.2. SOME CLASSES OF FUZZY SET UNIONS AND INTERSECTIONS.

Reference	Fuzzy Unions	Fuzzy Intersections	Range of Parameter
Schweizer & Sklar [1961]	$1 - \max[0, (1-a)^{-p} + (1-b)^{-p} - 1)]^{1/p}$	$\max(0, a^{-p} + b^{-p} - 1)^{-1/p}$	$p \in (-\infty, \infty)$
Hamacher [1978]	$\frac{a + b - (2 - \gamma)ab}{1 - (1 - \gamma)ab}$	$\frac{ab}{\gamma + (1 - \gamma)(a + b - ab)}$	$\gamma \in (0, \infty)$
Frank [1979]	$1 - \log_s \left[1 + \frac{(s^{1-a} - 1)(s^{1-b} - 1)}{s - 1}\right]$	$\log_s \left[1 + \frac{(s^a - 1)(s^b - 1)}{s - 1}\right]$	$s \in (0, \infty)$
Yager [1980]	$\min[1, (a^w + b^w)^{1/w}]$	$1 - \min[1, (1 - a)^w + (1 - b)^w)^{1/w}]$	$w \in (0, \infty)$
Dubois & Prade [1980]	$\frac{a + b - ab - \min(a, b, 1 - \alpha)}{\max(1 - a, 1 - b, \alpha)}$	$\frac{ab}{\max(a,b,\alpha)}$	$\alpha \in (0, 1)$
Dombi [1982]	$\frac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{-\lambda} + \left(\frac{1}{b} - 1\right)^{-\lambda}\right]^{-1/\lambda}}$	$\frac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{b} - 1\right)^{\lambda}\right]^{1/\lambda}}$	$\lambda \in (0, \infty)$

p 50 of Klir

Them: 2.8

$\forall\ (a,b) \in [0,1]\ ,\ u(a,b) \geq \max(a,b)$

Thus, $\max(a,b)$ = 'standard' union

is smallest fuzzy set that satisfies the skeletal axioms

Proof:

$u[a,0]=a$ ⟵ Lemma

$$u(b,0) \underset{\uparrow \atop \text{Commut}}{=} u(0,b) = b$$

From monotonicity

$$u(a,b) \geq u(a,0) = a$$

$$u(a,b) \geq u(0,b) = b$$

$$\Longrightarrow u(a,b) \geq \max(a,b)$$

QED

Them: 2.10

$\forall\ a,b \in [0,1]$, $i(a,b) \le \min(a,b)$

Thus, $\min(a,b)$ = 'standard' intersection is largest membership that satisfies skeletal axioms.

Proof:

$i(a,1) = a$ ← Lemma

$i(a,b) \le i(a,1) = a$ ← Monotonicity

$i(a,b) \le i(b,1) = b$

$\Longrightarrow i(a,b) \le \min(a,b)$

Theorem 2.12

$u(a,b) = \max(a,b)$ is <u>only</u> continuous and idempotent $(u(a,a)=a)$ fuzzy union that satisfies the skeletal union axioms. (wow!)

Proof:

$u(a, u(a,b)) = u(u(a,a), b) \Leftarrow$ Assoc

Assume idempotence:

Typo on p.55

(1) $u(a, u(a,b)) = u(a,b)$

Similarly:

(2) $u(u(a,b), b) = u(a, u(b,b)) = u(a,b)$

Equate 1 & 2:

$u(a, u(a,b)) = u(u(a,b), b)$

Commutative:

$u(a, u(a,b)) = u(b, u(a,b))$

Let $a < b$ (wloc) and $u(a,b) = \alpha \neq \left\{ \begin{matrix} a \\ b \end{matrix} \right.$

$\Rightarrow u(a,\alpha) = u(b,\alpha)$

But $\exists (a,b)$ $u(a,\alpha) < u(b,\alpha)$ by monotinicity & continuity

Thus $\alpha \neq a$ and $\alpha \neq b$ is bad assumption

$\therefore$, $\alpha = a$ or b

$\alpha = \max(a,b) = b$ works, since

$u(a,b) = u(b,b) = b$

$\alpha = \min(a,b) = a$ does not work.

$\therefore$, $\alpha = \max(a,b) = u(a,b)$ if:

Associative
i Idempotent
ii Commutative
iii monotonicity
iv continuity

Them 2.13

Skeletal intersection axioms
+ cont + indempotence

$$i(a,b) = \min(a,b)$$

is unique.

(Proof similar)

Law of Excluded Middle

$A \cup \bar{A}$ = UNIVERSAL SET

$\mu_{A \cup \bar{A}} = 1$

Law of Contradiction

$A \cap \bar{A} \neq \phi$

$\mu_{A \cap \bar{A}} = 0$

Theorem: (Skeletal axioms and continuity)

1. $A \cup \bar{A}$ = Universal Set $\Longrightarrow$ $A \cup A \neq A$
2. $A \cap \bar{A} = \phi$ $\Longrightarrow$ $A \cap A \neq A$

3. $A \cup \bar{A} = U$, $A \cap \bar{A} = \phi$ both imply $\cap$ & $\cup$ are not distributive

wow!

Proof (for law of excluded middle)

Contra of $A \cup \bar{A} = U \Rightarrow A \cup A \neq A$

is $A \cup A = A \Rightarrow A \cup \bar{A} \neq U$

Assume 1. Skeletal Axioms
2. Continuity
3. $A \cup A = A$ (Idempotence)

$\Rightarrow$ only solution for union is max

$\Rightarrow \mu_{A \cup A} = \max \Rightarrow \mu_{A \cup \bar{A}} \neq 1$

QED

Similar proof for law of contradiction

Second Part of Them:

If
1. Skeletal Axioms & Continuity
2. Excluded Middle $A \cup \bar{A} = U$
3. Contradiction $A \cap \bar{A} = \phi$

$\Longrightarrow$ Then Distributive laws do not work.

DISTRIBUTIVE LAWS

① $u(a, i(b,d)) = i(u(a,b), u(a,d))$ ⊛

$\Rightarrow A \cup (B \cap D) = (A \cup B) \cap (A \cup D)$

Recall equilibrium point of complementation:

$e = c(e) \Leftarrow$ Proved from assumption 1

Thus

$u(e, c(e)) = u(e,e) = 1 \Leftarrow$ 2 Excluded Middle

$i(e, c(e)) = i(e,e) = 0 \Leftarrow$ Contradiction

From ⊛

$u(e, i(e,e)) = u(e,0)$ ⊕ Boundary Condition

Note: ~~$e \neq 0$ since $C(0) = 1$ & $0 \neq 1$~~
$e \neq 1$ " $C(1) = 0$ & $1 \neq 0$

Recall proof from skeletal axioms:

$u(e,0) = e$

Thus ⊕ becomes

$u(e, i(e,e)) = e \neq 1$ ⊗ left side of ⊛

cont →

$a, b, d = e$ in right side of $\circledast$

$$i\left(u(e,e), u(e,e)\right)$$
$$= i\left(u(e,\bar{e}), u(e,\bar{e})\right)$$
$$= i(1,1) = 1 \quad \Leftarrow \text{Boundary condition}$$
$$\neq e \text{ in } \otimes$$

Similar proof for

$$i(a, u(b,d)) \neq u(i(a,b), i(a, d))$$

under same assumptions

In general, choose

Crisp Inclusive Or

Law of Excluded Middle Law of Contradiction	OR	Idempotency Distributivity
$\Downarrow$		$\Downarrow$
$u(a,b)=\min(1, a+b)$		$u(a,b)=\max(a,b)$
$i(a,b)=\max(0, a+b-1)$		$i(a,b)=\min(a,b)$
$c(a)=1-a$		$c(a)=1-a$

2.6. General Aggregation Operations (Vote)

$h: [0,1]^n \longrightarrow [0,1]$

Several fuzzy sets are combined into one set:

$$\mu_A(x) = h\left(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right)$$

(Ex: Everybody's idea of a fuzzy set is different).

Skeletal

Axiom h 1: Boundary Conditions

$$h(\vec{0}) = 0 \quad , \quad h(\vec{1}) = 1$$

Axiom h2: Monotonic nondecreasing

$$h(\vec{x} + \vec{\Delta}) \geq h(\vec{x})$$

when $\vec{\Delta}$ contains positive components

Axiom h3: h is Continuous

Axiom h 3: h is Symmetric

$$h(\vec{x}) = h(\vec{x}_p)$$

where $\vec{x}_p$ is a permutation of $\vec{x}$.

Averaging Operations

$$\min(\vec{a}) \le h(\vec{a}) \le \max(\vec{a})$$

Generalized means:

$$h_\alpha(\vec{a}) = \left(\frac{1}{n}\sum_{k=1}^{n} a_n^\alpha\right)^{\frac{1}{\alpha}} \quad -\infty<\alpha<\infty$$

Special Cases

$$h_{-\infty}(\vec{a}) = \min(\vec{a})$$

$$h_\infty(\vec{a}) = \max(\vec{a})$$

$$h_0(a) = \left(\prod_{n=1}^{N} a_n\right)^{1/N} \Leftarrow \text{Geometric Mean}$$

(proof using l'Hospital)

$$h_{-1}(\vec{a}) = N\sum_{n=1}^{N}\frac{1}{a_n} \Leftarrow \text{harmonic mean}$$

$$h_1(\vec{a}) = \frac{1}{N}\sum_{n=1}^{N} a_n \Leftarrow \text{arithmetic mean}$$

CHAPTER 3

FUZZY RELATIONS

3.1. Crisp & Fuzzy Relations

Cartesian Product 'twixt Crisp Sets $\mathbb{X}$ & $\mathbb{Y}$

$$= \mathbb{X} \times \mathbb{Y} = \{(x,y) \mid x \in \mathbb{X}, y \in \mathbb{Y}\}$$

$$\neq \mathbb{Y} \times \mathbb{X} \text{ in general}$$

Extention

$$\underset{i \in \mathbb{N}_n}{\times} \mathbb{X}_n = \{(x_1, \dots, x_n) \mid x_m \in \mathbb{X}_m; 1 \le m \le n\}$$

$R(\mathbb{X}_1, \dots, \mathbb{X}_n) =$ <u>relation</u> among sets

$$\subset \mathbb{X}_1 \times \dots \times \mathbb{X}_n$$

Example: 2 coins:

$$\mathbb{X} = \{0, 1\}$$

$$\mathbb{Y} = \{0, 1)$$

$$\mathbb{X} \times \mathbb{Y} = \begin{Bmatrix} (1,0) & (1,1) \\ (0,0) & (0,1) \end{Bmatrix}$$

$R(\mathbb{X}, \mathbb{Y}\} =$ sum is 1

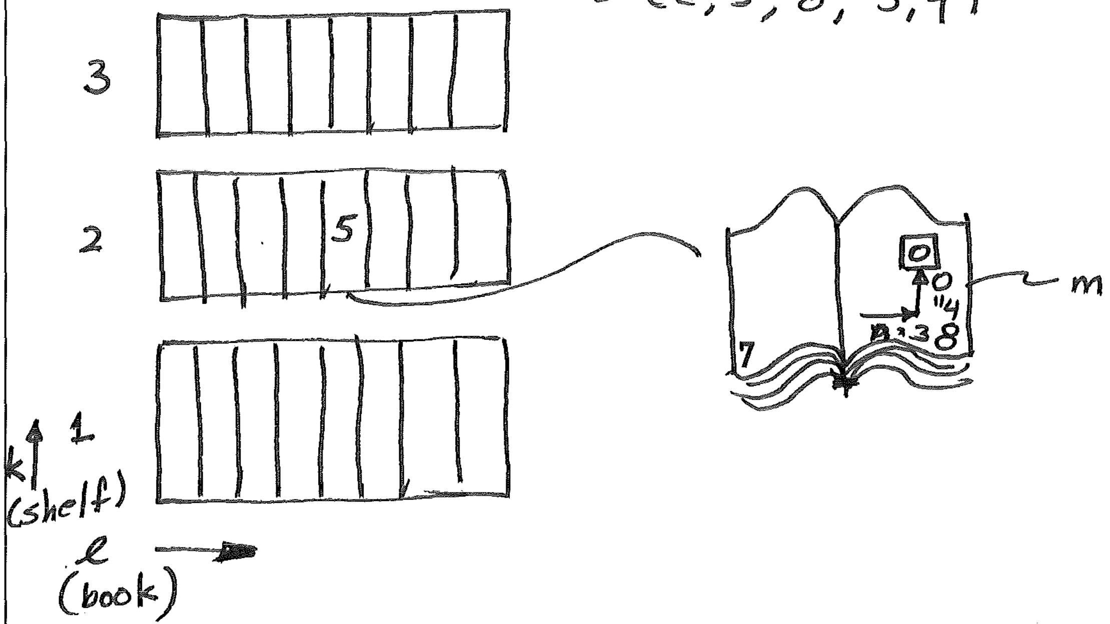
$$= \{(1,0), (0,1)\} \subset \mathbb{X} \times \mathbb{Y}$$

Crisp Membership Set:

$$\mu_R(x_1,\dots,x_n) = \begin{cases} 1 & ;\text{ iff } (x_1,\dots,x_n)\in R \\ 0 & ;\text{ o.w.} \end{cases}$$

n-d arrays can alway be decomposed to 2-D arrays.

Example $\vec{x} = (k, \ell, m, n, o)$
$= (2, 5, 8, 3, 4)$



Example: N-D Tic-Tac-Toe

FUZZY RELATION: A fuzzy set defined on $\mathbb{X}_1 \times \ldots \times \mathbb{X}_n \ni \vec{x} \in$ may have varying degrees of membership.

Example:

‘very far’		$\mathbb{X}$	
		NYC	Paris
	Beijing	1	0.9
$\mathbb{Y}$	NYC	0	.7
	London	.6	.3

or

$R(\mathbb{X},\mathbb{Y}) =$ 1/NYC, Beijin
+ 0/NYC, NYC
+ 0.6/NYC, London
+ 0.9/Paris, Beij
+ 0.7/ " , NYC
+ 0.3/ " , London

Subsequence:

$\vec{y} \prec \vec{x} \Rightarrow y$ is a subsequence of x

Example

	Paris
Beijing	0.9
NYC	0.7

<u>OR</u>

0.1/ Paris, Beijing
+ 0.7 Paris, NYC   subsequence

'PROJECTION'

$[R \downarrow \underline{Y}]$ = 'projection of the set R onto set $\underline{Y}$'

$$\mu_{[R \downarrow \underline{Y}]}(y) = \max_{x \succ y} \mu_R(x)$$

$\uparrow$

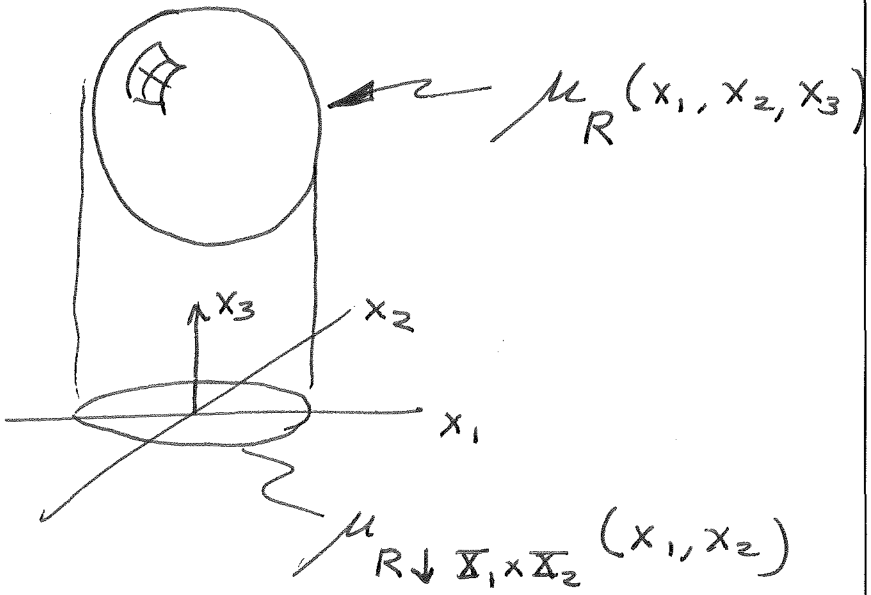
maximum over all subsequences

$$\begin{array}{c|cc} & NYC & P \\ \hline B & 1 & 0.9 \\ NYC & 0 & 0.7 \\ L & 0.6 & 0.3 \end{array}$$

CITIES FAR FROM NYC or PARIS

$$\begin{array}{c|c} B & 1 \\ NYC & 0.7 \\ L & 0.6 \end{array}$$

Note: Projection is zadeh's shadow:



$$= \max_{x_3} \mu_R(x_1, x_2, x_3)$$

R: Condiments, High Protein Food & Spread Good on Bread

X_1 = {Mayo, (Grape) Jelly} ← COND

X_2 = {Ham, PNB} ← FOOD

X_3 = {Butter, Cream Cheese} ← SPREAD

~~R = tastes~~ good

$R(X_1, X_2, X_3)$

= 0.7 / mayo, ham, butter
+ 0.5 / mayo, ham, cream cheese
+ 0.7 / mayo, pnb, butter
+ 0.3 / mayo, pnp, cream cheese
+ 0 / jelly, ham, butter
+ 0.1 / jelly, ham, cream cheese
+ 0.8 / jelly, pnb, butter
+ 0.4 / jelly, pnb, cream cheese

Project Outputter ; $\underline{X} = \underline{X}_1 \times \underline{X}_2 \times \underline{X}_3$

$\mu\, R \downarrow \underline{X} - \underline{X}_3 = {}^{\mu}R \downarrow \underline{X}_1 \times \underline{X}_2$

= max (0.7, 0.5) / mayo, ham
+ max (0.7, 0.3) / mayo, pnb
+ max (0, 0.1) / jelly, ham
+ max (0.8, 0.4) / jelly, pnb

* = 0.7 / mayo, ham
+ 0.7 / mayo, pnb
+ 0 / jelly, ham
+ 0.8 / jelly, pnb

= Condiments & High Protein
Food that go together on bread

Project out condiments

$\mu\, R \downarrow (\underline{X} - \underline{X}_3) - \underline{X}_1$
$= {}^{\mu}R \downarrow \underline{X}_2$
= max (0.7, 0) / ham
+ max (0.7, 0.8) / pnb
} High protein foods that go together on bread.
= 0.7 / ham
+ 0.8 / pnb

OR interpretation

ham tastes good on bredd
= mayo / ham taste good on bread
OR jelly, ham " " " "
max, or can use other fuzzy union

Extending a subsequence to a sequence.

~~1. Cyclic extention~~

~~1. Cyly~~

1. Cylindric Extention
2. " Closure

FIRST ⇒ Cylindric Extention

~~$R \uparrow \mathbf{X} - \mathbf{Y}$~~

Before:

Projection:

$$\mu_{[R \downarrow \mathbf{Y}]}(\vec{y}) = \max_{x \succ y} \mu_R(\vec{x})$$

Cylindric Extention

$$\mu_{[R \uparrow \mathbf{X} - \mathbf{Y}]}(x) = \mu_R(y) \;\; \forall \;\; x \succ y$$

Example:

Recall:

$\hat{R} = R \downarrow X_2 = 0.7/ham + 0.8/pnb$

~~$(R \downarrow X_2) \uparrow$~~ X_2 = condiments

$\mu_R(\hat{R} \uparrow X_2) =$ 0.7 / ~~ham~~, mayo, ham
+ 0.7 / jelly, ham
+ 0.8 / mayo, pnb
+ 0.8 / jelly, pnb

~~Cyclic~~ Cylindric <u>extention</u> gives largest fuzzy set that castes right shadow.

⇒ Is 'least' specific
'maximimizes the nonspecificity'

2. Cylindrical Closure

Estimates from two or more shadows (min-max tomography).

$$\mu_{cyl\{R_i\}}(\vec{x}) = \min \underbrace{\mu_{R \uparrow \mathbb{X} - \mathbb{Y}_i}(\vec{x})}_{\text{cylindric extention}}$$

Example:

$\mu_{\mathbb{X}_2} = 0.7/ham + 0.8/pnb$

$\mu_{\mathbb{X}_1} = max(0.7, 0.7)/mayo + max(0, 0.8)/jelly$

$= 0.7/mayo + 0.8/jelly$

CONDIMENT

From * on p 66 of notes

Two Cylindric Extentions

$\mu_{\mathbb{X}_2}$

	ham 0.7	pnb 0.8
mayo	0.7	0.8
jelly	0.7	0.8

$\mu_{\mathbb{X}_1}$

mayo 0.7	0.7	0.7
jelly 0.8	0.8	0.7

Largest fuzzy set ~~contain~~ that will give all projections.

Run a min on above 2

$$\begin{array}{c|cc} mayo & 0.7 & 0.7 \\ jelly & 0.7 & 0.8 \end{array}$$

Cylindrical Closure

Real result:

$$\begin{array}{c|cc} & ham & pnb \\ \hline mayo & 0.7 & 0.8 \\ jelly & 0.7 & 0.8 \end{array}$$

(Close!)

An interpretation of cylindrical closure

$\underline{X}$ = men who are tall

$R(\underline{X}) = 0.2/5' + 0.5/6' + 0.9/7'$

$\underline{Y}$ = mem w/ big feet

$R(\underline{Y}) = 0.2 / \text{Size } 11 + 0.5 / \text{size } 13 + 0.9 / \text{Size } 15$

$\underline{X} \times \underline{Y}$ = men who are tall <u>and</u> have big feet

$\Rightarrow$ Do cyclindrical closure

note: 'min' outer product

			Tall	
		5'	6'	7'
		0.2	0.5	0.9
	⑪ 0.2	.2	.2	.2
Big Feet	⑬ 0.5	.2	0.5	0.5
	⑮ 0.9	.2	0.5	0.9

Analogy:

Marginals: $f(x_1)$, $f(x_2)$

Joint Pdf estimate: $f(x_1) f(x_2)$

$\mathrel{\hat{=}} p(x_1, x_2)$

Note:

There is an assumption of existance

$R(X) = 0.1/4' + 0.2/5' + 0.3/5'3''$

$R(Y) = 0.2/11 + 0.5/13 + 0.9/15$

	Tall		
	0.1	0.2	0.3
0.2	0.1	0.2	0.2
0.5	0.1	0.2	0.3
0.9	0.1	0.2	0.3

} Does **not** give marginals (nothing will)

^ **BUT**

3.2. BINARY RELATIONS

between sets $\mathbb{X}$ & $\mathbb{Y}$ = $R(\mathbb{X},\mathbb{Y})$

⟹ CRISP SETS

★ DOMAIN = SET OF ALL POINTS in X which participate in the relation

$$\text{dom } R(\mathbb{X},\mathbb{Y}) = \{x \mid x \in \mathbb{X}, (x,y) \in R \text{ for some } y \in \mathbb{Y}\}$$

★ RANK = SET OF ALL POINTS IN Y which participate in the relation.

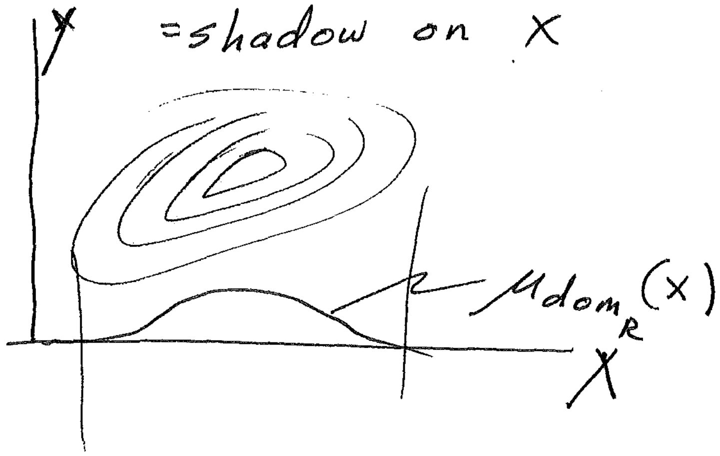
$$\text{ran } R(\mathbb{X},\mathbb{Y}\} = \{y \mid y \in \mathbb{Y}, (x,y) \in R \text{ for some } x \in \mathbb{X}\}$$

⟹ FUZZY ~~SET~~ RELATIONS

★ DOMAIN = FUZZY SET OF POINTS IN X which participate in the fuzzy relation. The degree of membership ~~is~~ in the domain set ≡ maximum to the contribution.

$$\mu_{\text{dom } R}(x) = \max_{y \in \mathbb{Y}} \mu_R(x,y)$$

= shadow on X



☆ rank $R(x,y)$ is dual

= shadow of $\mu_R(x,y)$ on Y

$$\Rightarrow \mu_{\text{ran}\,R}(y) = \max_{x \in X} \mu_R(x,y)$$

The 'height' of a fuzzy relation R is

$$h(R) = \max_{x \in X} \max_{y \in Y} \mu_R(x,y)$$

$h(R) = 1 \Rightarrow$ normal

$h(R) < 1 \Rightarrow$ subnormal

Mapping: Each member in domain of R ~~is~~ appears exactly once in R

$$\begin{array}{cc} \uparrow y & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/5 \end{bmatrix} \leftarrow \text{mapping} \\ & x \longrightarrow \end{array}$$

One to One: Eech element of range appears exactly once in the mapping

$$\begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{One to One}$$

Go to photocopy on p. 22C of notes.

$h(R) = 1$

not a mapping
↑
~~one to~~
~~0~~

α cut of R same as for fuzzy set

$$R = \begin{bmatrix} 1 & 0.5 \\ 0.2 & 0.7 \end{bmatrix} \xrightarrow{\alpha = 0.4} R_{0.4} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$R_\alpha = \alpha \text{ cut of } R \Rightarrow \mu_{R_\alpha} = \overset{\text{UNIT STEP}}{U}(\mu_R - \alpha)$$

Every fuzzy relation can be expressed by:

$$R = \bigcup_\alpha \alpha R_\alpha$$

$$\mu_{\alpha R_\alpha} = \alpha \mu_R$$

For example at top of page,

$$\alpha R_\alpha = \begin{bmatrix} 0.4 & 0.4 \\ 0 & 0.4 \end{bmatrix} \quad ; \alpha = 0.4$$

Example:

$$\begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.0 \\ 0.2 & 0.4 \end{bmatrix} = 0.1 \times \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \overset{U}{+} 0.2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \overset{U}{+} 0.4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

U = max OPERATOR

$\downarrow$

Ra ?

Inverse

$X \times Y$ has inverse $Y \times X$

$R^{-1}(\underset{\sim}{X}, \underset{\sim}{Y}) = \{ (y, x) \mid (x, y) \in R \}$

For fuzzy relations:

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y) \quad \text{(Transpose)}$$

M_R = matrix representation of R

$M_{R^{-1}} = M_R^T$

COMPOSITION:

$$R(\underline{X}, Z) = P(X, Y) \circ Q(Y, Z)$$

= min max matrix multipication covered in Zadeh's paper

$$\mu_{P \circ Q}(x,z) = \max_{y} \min [\mu_P(x,y), \mu_Q(y,z)]$$

If:

$$M_{P \circ Q} = M_P \circ M_Q$$

$$\begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.4 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$$

$$\neq M_Q \circ M_P$$

BUT:

$$M_{P \circ Q}^{T} = (M_P \circ M_Q)^{T}$$

$$= M_Q^{T} \circ M_P^{T} \quad ; \quad \text{Recall } (^{T}) \text{ is inverse.}$$

$$[0.1 \quad 0.3 \quad 0.1] = [0.3 \quad 0.1] \begin{bmatrix} 0.1 & 0.4 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}$$

OF Course!

EXACT SAME OPERATIONS

Alternate

(Use product as 'fuzzy min')

$$R(X,Z) = P(X,Y) \odot Q(Y,Z)$$

$$\mu_{P \odot Q}(x;z) = \max_{y} \; u_P(x,y) \cdot \mu_Q(y,z)$$

$$M_{P \odot Q} = M_P \odot M_Q$$

$$(M_{P \odot Q})^T = M_Q^T \odot M_P^T$$

$$\begin{bmatrix} 0.1 & 0 \\ 0.4 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \odot \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix} = \begin{bmatrix} \max(0.03, 0) \\ \max(0.12, 0.01) \\ \max(0.03, 0.02) \end{bmatrix}$$

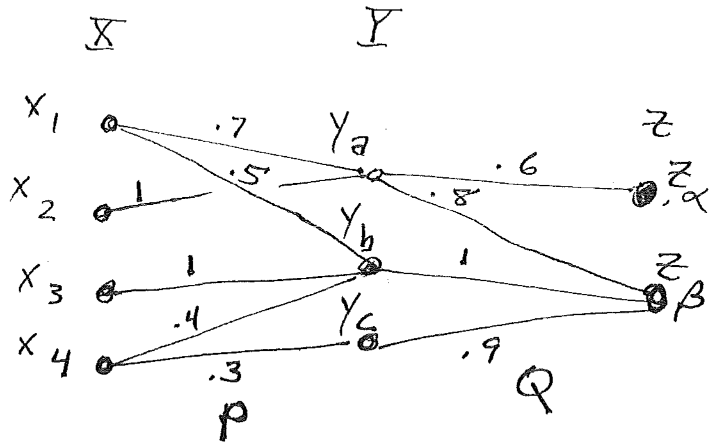
$$= \begin{bmatrix} 0.03 \\ 0.12 \\ 0.03 \end{bmatrix}$$

Relational Join

$$P(X, Y) * Q(Y, Z)$$
$$= \{(x,y,z) \mid (x,y) \in P \text{ and } (y,z) \in Q\}$$

$$\mu_{P*Q}(x,y,z) = \min[\mu_P(x,y), \mu_Q(y,z)]$$

Ex

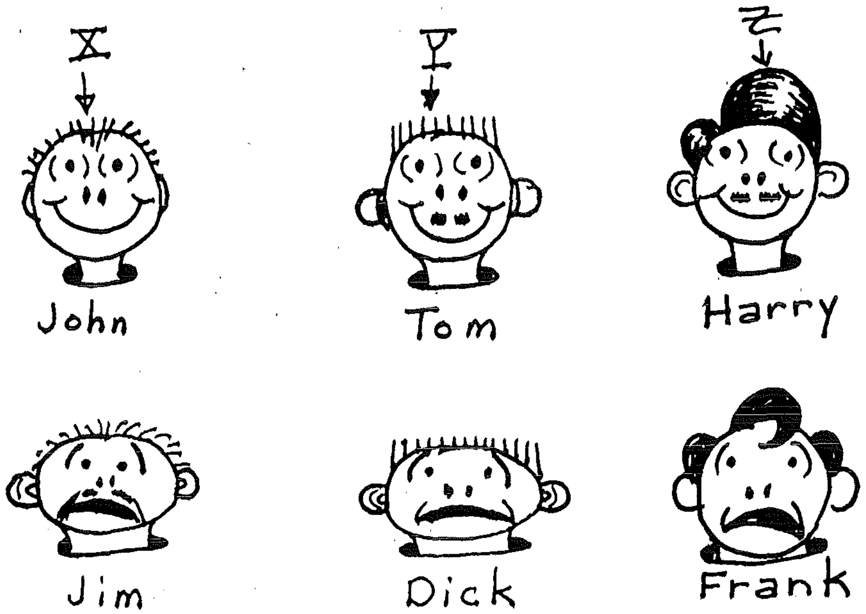


← sagittal diagram

$$\mu_{P*Q}(x_1, y_a, z_\alpha) = \min[0.7, 0.6] = 0.6$$

$$M_{P*Q} = \begin{array}{c|ccc|ccc|} & & z_\alpha & & & z_\beta & \\ & y_a & y_b & y_c & y_a & y_b & y_c \\ x_1 & 0.6 & 0 & 0 & .7 & .5 & 0 \\ x_2 & .6 & 0 & 0 & .5 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_4 & 0 & 0 & .3 & 0 & .4 & .3 \end{array}$$

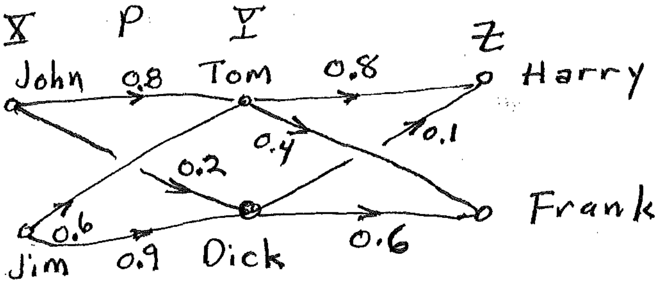
Join ~ outer product



Recall:

$$P(X \times Y) \qquad\qquad\qquad Q(Y \times Z)$$

$$\underbrace{\begin{array}{c|cc} & \text{John} & \text{Jim} \\ \hline \text{Tom} & 0.8 & 0.6 \\ \text{Dick} & 0.2 & 0.9 \end{array}}_{M_P} \qquad \underbrace{\begin{array}{c|cc} & \text{Harry} & \text{Frank} \\ \hline \text{Tom} & 0.8 & 0.4 \\ \text{Dick} & 0.1 & 0.6 \end{array}}_{M_Q}$$



M_{P*Q} =

<u>Harry</u>

$$\begin{array}{lcc} & \text{Tom} & \text{Dick} \\ \text{John} & 0.8 & \enclose{circle}{0.1} \\ \text{Jim} & 0.6 & 0.6 \end{array}$$

<u>Frank</u> ⇐ Join

$$\begin{array}{lcc} & \text{Tom} & \text{Dick} \\ \text{John} & 0.4 & 0.2 \\ \text{Jim} & 0.4 & 0.6 \end{array}$$

Resemblance of John, Dick, Harry

= " of John & Dick

<u>and</u> the resemblance of Dick & Harry

= min (0.2, 0.1) = 0.1

Contrast <u>Join</u> to <u>Composition</u>

$$\begin{array}{lcc} & \text{Tom} & \text{Dick} \\ \text{John} & 0.8 & 0.2 \\ \text{Jim} & 0.6 & 0.9 \end{array} \qquad \begin{array}{lcc} & \text{Harry} & \text{Frank} \\ \text{Tom} & 0.8 & 0.4 \\ \text{Dick} & 0.1 & 0.6 \end{array}$$

$$M_{P\circ Q} = \begin{array}{lcc} & \text{Harry} & \text{Frank} \\ \text{JOHN} & 0.8 & 0.4 \\ \text{Jim} & 0.6 & 0.6 \end{array}$$

What is resemblance of Tom & Dick?

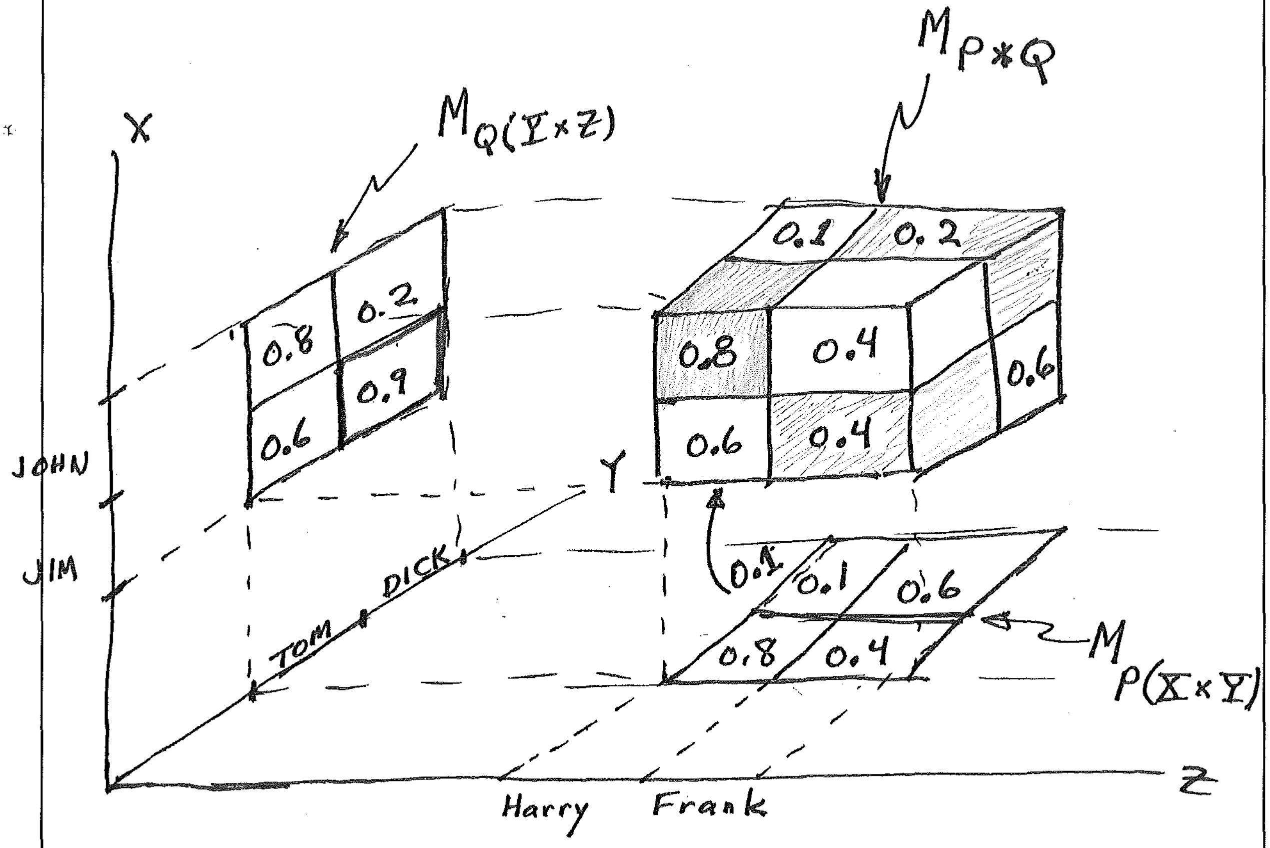
⇒ Res(Tom, John and John, Dick)

<u>OR</u> Res (Tom, Jim and Jim, Dick)

= max[min(0.8, 0.2), min(0.6, 0.9)]

= 0.6

Similarity
to Cylindrical Closure



Q: Does
M_{P*Q} castes shadows of
M_P & M_Q ?

A: NO

CYLINDRICAL CLOSURE DOES!

Difference: Common Y ←

3.3. BINARY RELATIONS ON A SINGLE SET

$X = \{\text{Beijing}, \text{NY}, \text{Miami}\}$

$R =$ 'far'

$$\begin{array}{cccc} & B & N & M \\ B & 0 & 1 & 0.9 \\ N & 1 & 0 & 0.3 \\ M & 0.9 & 0.3 & 0 \end{array} \leftarrow R(X \times X)$$

Properties:

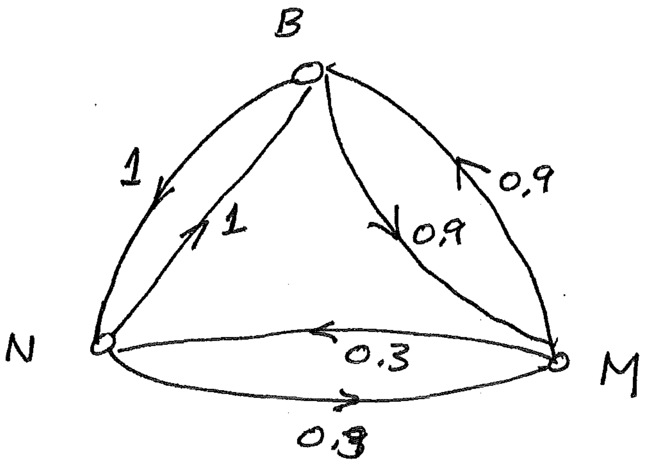
1. Symmetry

$$M_R = M_R^T$$

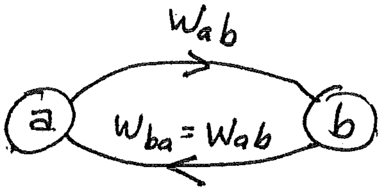
OR $\mu_R(x,y) = \mu_R(y,x)$

Symmetric

DIAGRAM



Symmetry in general:



2 TRANSITIVE

$(x, y) \in R,\ (y, z) \in R \Rightarrow (x, z) \in R$

Example (Crisp)

$\mathbf{X}$ = courses

$\{EE335,\ EE505,\ EE595\}$

$R(\mathbf{X} \times \mathbf{X}\}$

prerequisites

for this

Courses

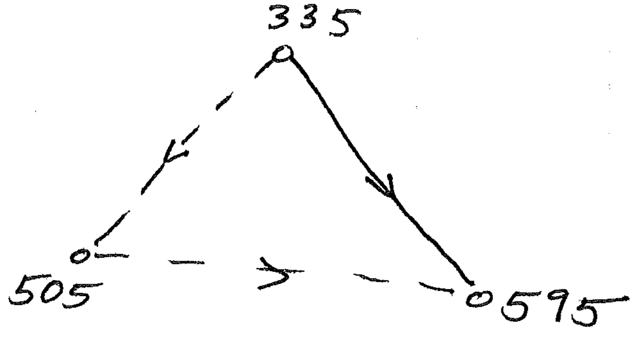
$$\begin{array}{cccc} & 595 & 505 & 335 \\ 595 & 0 & 0 & 0 \\ 505 & 1 & 0 & 0 \\ 335 & 1 & 1 & 0 \end{array}$$

Pre

$(335, 505) \in R,\quad (505, 595) \in R$

$\Rightarrow (335, 595) \in R$

Diagram:



If $- - \rightarrow - -$

Then $\longrightarrow$

FUZZY TRANSITIVE:

$$\mu_R(x,z) > \max_{y} \min \mu_R(x,y), \mu_R(y,z)$$

$$\forall \; (x,z) \in \mathbb{X}^2$$

$R(\mathbb{X},\mathbb{X}) =$ Importance of material in prerequisite

Course

	595	505	335
595	0	–	–
505	0.5	0	–
335	0.8	0.6	0

Pre

$$\mu_R(335,595) = 0.8$$

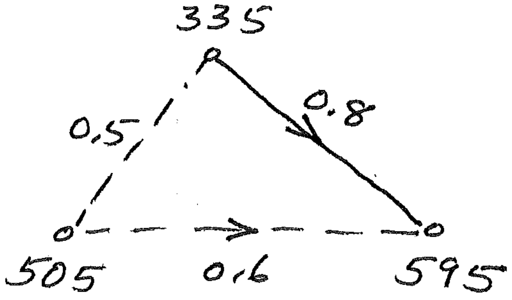
$$> \max \min \mu_R(335,505), \mu_R(505,595)$$

ONLY ONE ENTRY. NOT USED HERE

$$= \min(0.5, 0.6)$$

$$= 0.5$$

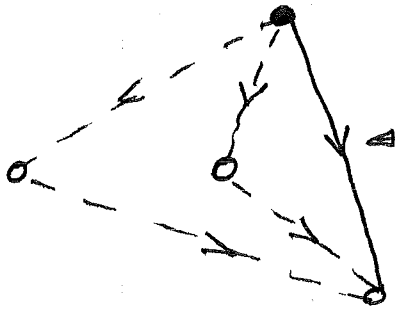
Diagram



Path →

≥ min of the two - -→- Paths

More general

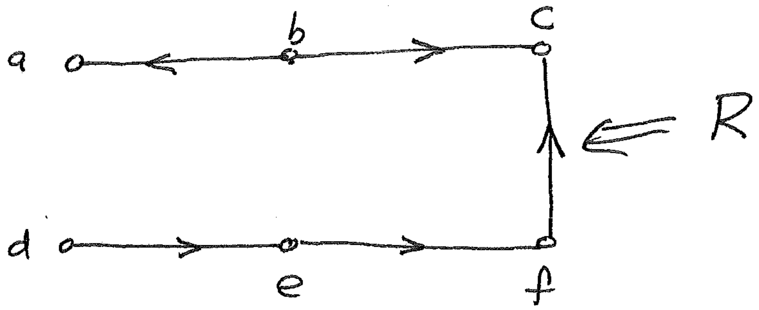


This path

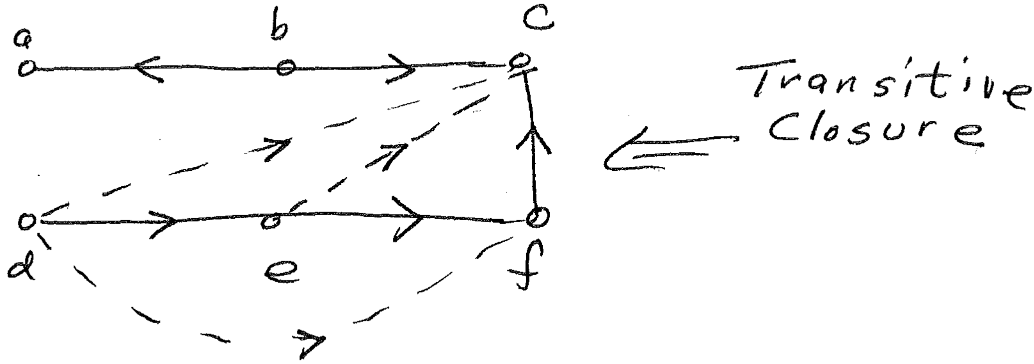
≥ max min of
other two
paths

(Crisp) Transitive Closure of $R(\underline{X},\underline{X})$
= the relation that is
transitive, contains R, and
has the fewest possible members.

Example (Using graphs)



Transitive Closure

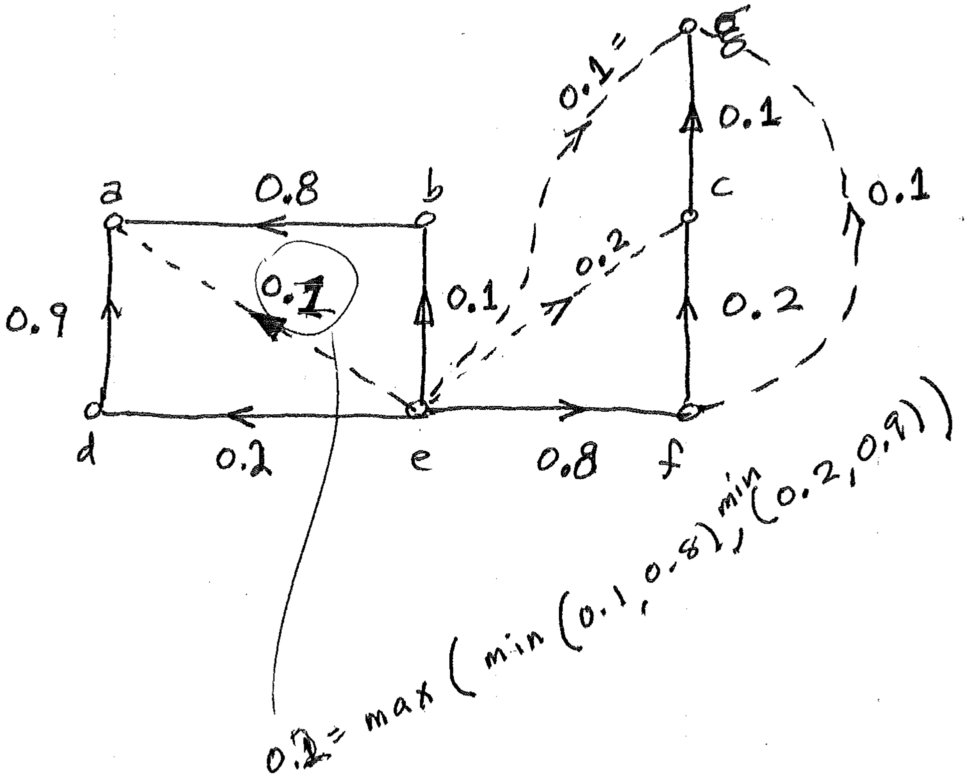


Fuzzification of Transitive Closure

= The relation that is transitive in the fuzzy sense that has the smallest membership values and contains R

Ex. ○→○ ≡ R

○⇢○ ≡ Closure(R)



$0.2 = \max(\min(0.1, 0.8), \min(0.2, 0.9))$

Algorithm to Find

R_T = TRANSITIVE CLOSURE OF R

① $n=0, \quad R_0 = R$

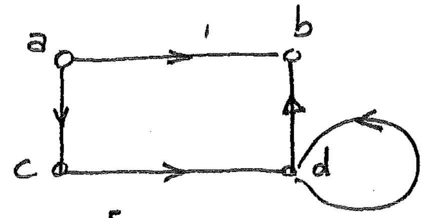
② $R_{n+1} = R_n \cup (R_n \overset{\text{composition}}{\circ} R_n)$

③ $R_{n+1} \neq R_n \Longrightarrow n = n+1$, GOTO STEP 2

else, $R_T = R_n$, STOP

Good for crisp & fuzzy

Ex:



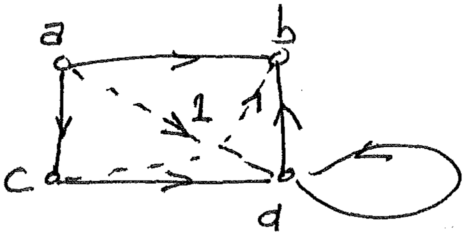
$$M_R = M_{R_0} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{R_0 \circ R_0} = M_{R_0} \circ M_{R_0} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

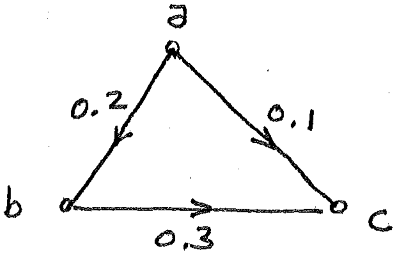
$$M_{R_1} = M_{(R_0 \circ R_0) \cup R_0} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{R_1 \circ R_1} = M_{R_1} \circ M_{R_1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{R_2} = M_{R_1 \cup (R_1 \circ R_1)} = \begin{bmatrix} 0 & 1 & 1 & \overset{\text{new}}{\textcircled{1}} \\ 0 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = M_{R_1} = M_{R_T}$$



Fuzzy Example:



$$M_R = M_{R_0} = \begin{bmatrix} 0 & .2 & .1 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix}$$

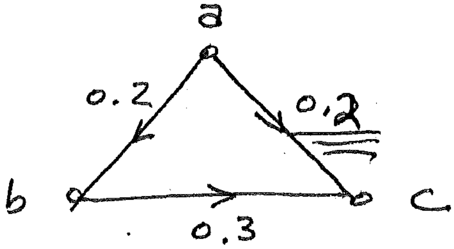
$$M_{R_0 \circ R_0} = M_{R_0} \circ M_{R_0}$$

$$= \begin{bmatrix} 0 & .2 & .1 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & .2 & .1 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & .2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_1} = M_{R_0 \cup (R_0 \circ R_0)} = \begin{bmatrix} 0 & .2 & .2 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix} \neq M_{R_0}$$

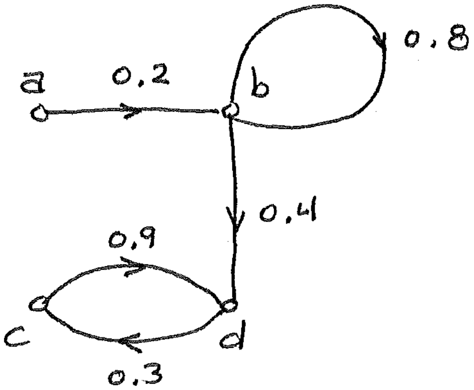
$$M_{R_1 \circ R_1} = \begin{bmatrix} 0 & .2 & .2 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & .2 & .2 \\ 0 & 0 & .3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & .2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \circ R_1 \quad \subset R_1 \Rightarrow \quad R_2 = R_1$$



⇐ TRANSITIVE CLOSURE

Another Fuzzy Example.



$$M_R = M_{R_0} = \begin{bmatrix} 0 & .2 & 0 & 0 \\ 0 & .8 & 0 & .4 \\ 0 & 0 & 0 & .9 \\ 0 & 0 & .3 & 0 \end{bmatrix}$$

$$M_{R_0 \circ R_0} = M_{R_0} \circ M_{R_0}$$

$$= \begin{bmatrix} 0 & .2 & 0 & 0 \\ 0 & .8 & 0 & .4 \\ 0 & 0 & 0 & .9 \\ 0 & 0 & .3 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & .2 & 0 & 0 \\ 0 & .8 & 0 & .4 \\ 0 & 0 & 0 & .9 \\ 0 & 0 & .3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & .2 & 0 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & .3 \end{bmatrix}$$

$$M_{R_1} = M_{R_0 \cup (R_0 \circ R_0)}$$

$$= \begin{bmatrix} 0 & .2 & 0 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .9 \\ 0 & 0 & .3 & .3 \end{bmatrix} \neq M_{R_0}$$

$$M_{R_1 \circ R_1} = M_{R_1} \circ M_{R_0}$$

$$= \begin{bmatrix} 0 & .2 & 0 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .9 \\ 0 & 0 & .3 & .3 \end{bmatrix} \circ \begin{bmatrix} 0 & .2 & 0 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .9 \\ 0 & 0 & .3 & .3 \end{bmatrix} = \begin{bmatrix} 0 & .2 & .2 & .2 \\ 0 & .8 & .3 & .3 \\ 0 & 0 & .3 & .3 \\ 0 & 0 & .3 & .3 \end{bmatrix}$$

R

$$M_{R_2} = M_{R_1 \cup (R_1 \circ R_1)} = \begin{bmatrix} 0 & .2 & \boxed{.2} & \boxed{.2} \\ 0 & .8 & \boxed{.3} & .4 \\ 0 & 0 & \boxed{.3} & .9 \\ 0 & 0 & .3 & \boxed{.3} \end{bmatrix} \neq M_{R_1}$$

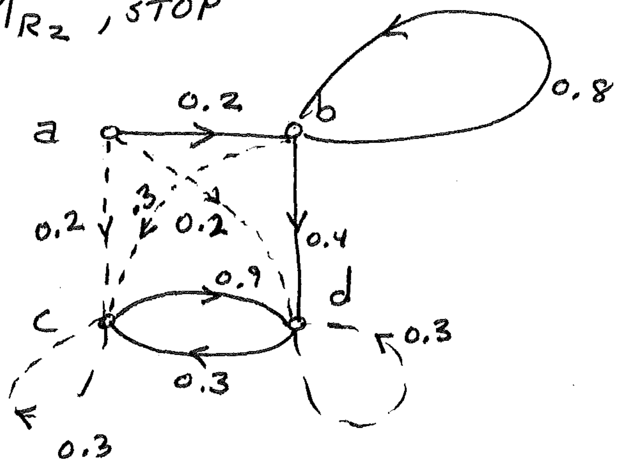
$$M_{R_2} \circ M_{R_2} = M_{R_2 \circ R_2} =$$

$$= \begin{bmatrix} 0 & .2 & .2 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .9 \\ 0 & 0 & .3 & .3 \end{bmatrix} \circ \begin{bmatrix} 0 & .2 & .2 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .9 \\ 0 & 0 & .3 & .3 \end{bmatrix} = \begin{bmatrix} 0 & .2 & .2 & .2 \\ 0 & .8 & .3 & .4 \\ 0 & 0 & .3 & .3 \\ 0 & 0 & .3 & .3 \end{bmatrix}$$

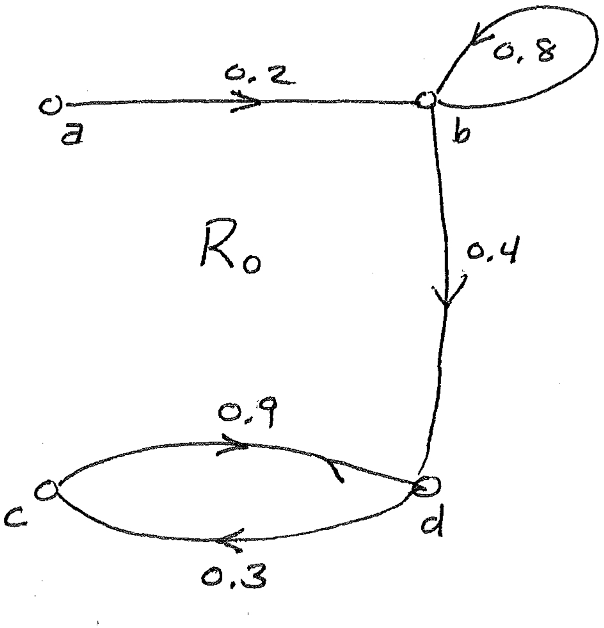
$$M_{R_2 \circ R_2} \subset M_{R_2} \Rightarrow M_{R_3} = M_{(R_2 \circ R_2) \cup R_2}$$

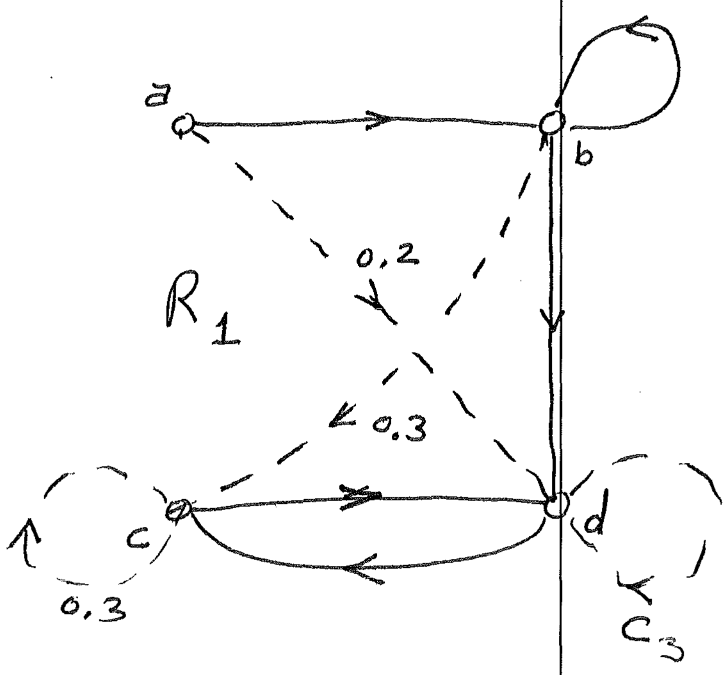
$$= M_{R_2 \circ R_2}$$

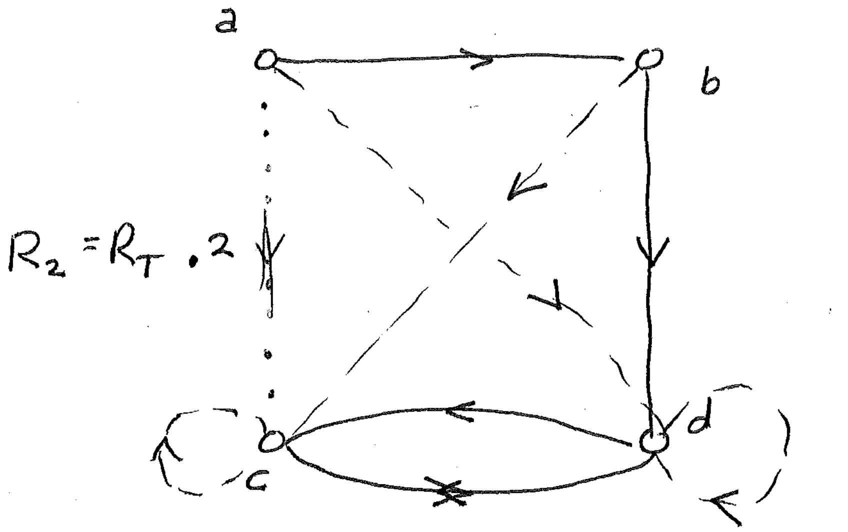
Since $M_{R_3} = M_{R_2}$, STOP



Convergence Dynamics:







Each iterations closes transition of previous graph. New paths may pose new transitions which are addressed in the next iteration, etc.

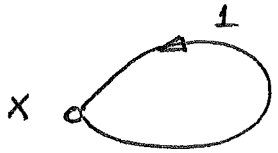
Properties of Binary Relations on a Single Set

1. Reflexive

a. DEF $\mathrm{diag}(M_R) = [1, \dots 1]^T$

or $\mu_R(x, x) = 1 \quad \forall \quad x \in \underline{X}$

b. diagram:



c. variations

- antireflexive $\exists \; x \ni \mu_R(x, x) \neq 1$

- $\in$ reflexitivity

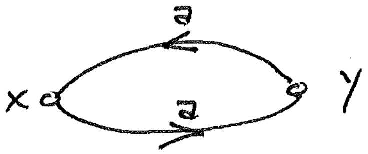
  $\mu_R(x, x) \geq \in$ for some $0 < \in < 1$

2. Symmetric

a. Def $M_R = M_R^T$

or $\mu_R(x, y) = \mu_R(y, x)$

b. Diagram



c. variations

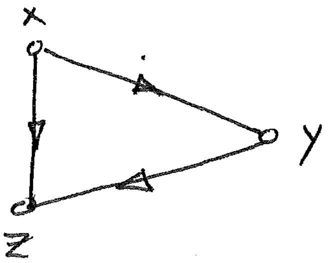
- asymmetric: $\exists \; (x, y) \ni \mu_R(x, y) \neq \mu_R(y, x)$

3. TRANSITIVE

a. Def $M_R = M_{R \cup (R \circ R)}$

$$\mu_R(x,z) \geq \max_y \min \mu_R(x,y), \mu_R(y,z)$$

b. diagram



c. Variations

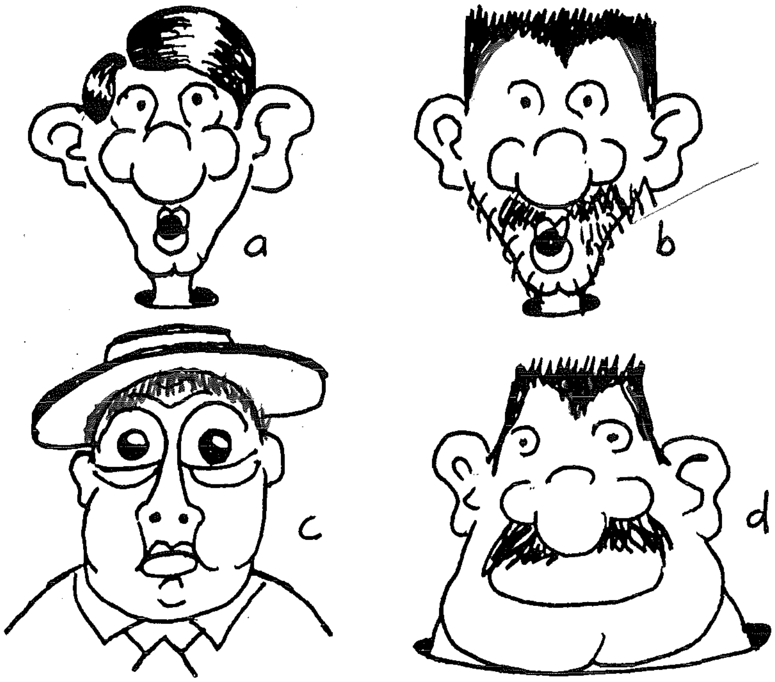
$\overline{\text{Transitive}}$ = non transitive

Antitransitive

$$\mu_R(x,z) < \max_y \min \mu_R(x,y), \mu_R(y,z)$$

Strict
Inequality

Example: Resemblance



An equivalence relation (crisp)

fuzzy similarity

1. Reflexive?

$$\mu_R(a,a) = 1 \Leftarrow \text{yes!}$$

2. Symmetric

$$\mu_R(a,b) = \mu_R(b,a) \Leftarrow \text{YES!}$$

3. Transitive

$$\mu_R(a,d) \geq \max\left[\min\left(\mu_R(a,b), \mu_R(b,d)\right), \min\left(\mu_R(a,c), \mu_R(c,d)\right]\right. ?$$

YES!

3.4. EQUIVALENCE AND SIMILARITY RELATIONS

CRISP

$$R\left\{\begin{matrix}\text{reflexive}\\ \text{symmetric}\\ \text{transitive}\end{matrix}\right\}\longrightarrow R \text{ is ‘equivalence relation’}$$

Formally:

$$A_x = \{y \mid (x,y) \in R(\underline{X},\underline{X})\}$$

X/R = family of all equivalence classes.

Example: $\underline{X} = \{a, b, c, d\}$

$$M_R = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \end{array} \Longleftarrow \text{equivalence relation}$$

$$\underline{X}/R = \{(a, c, d\}, \{b\}\}$$

Fuzzy R

$$R \rightarrow \left\{ \begin{array}{l} \text{reflexive} \\ \text{symmetric} \\ \text{transitive} \end{array} \right\} \longrightarrow \begin{array}{l} \text{(fuzzy)} \\ \text{similarity} \\ \text{relation} \end{array} \sim \begin{array}{l} \text{fuzzy} \\ \text{equivalence} \end{array}$$

Each α cut of a fuzzy relation has the form of an equivalence relation.

Example

$$M_R = \begin{matrix} 1 & .3 & .2 & .4 \\ .3 & 1 & .2 & .3 \\ .2 & .2 & 1 & .2 \\ .4 & .3 & .2 & 1 \end{matrix}$$

$$M_{R_{0.2}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \{\{a,b,c,d\}\}$$

are similar at level $\alpha = 0.2$

$$M_{R_{0.3}} = \begin{array}{cccc} a & b & c & d \\ \hline 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}$$

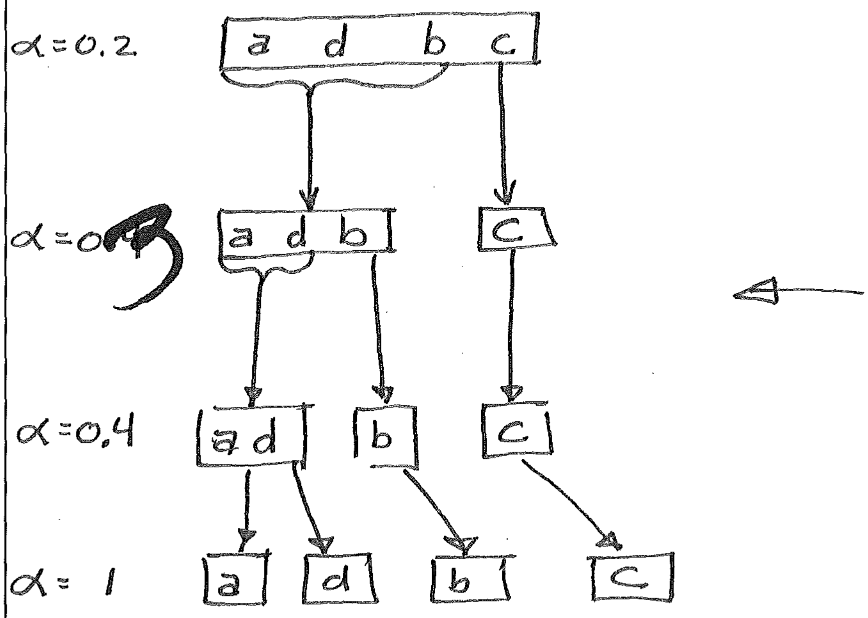
$$\underline{X}/R_{0.3} = \{ (a,b,d\}, \{c\}\} \Leftarrow \alpha = 0.3$$

$$M_{R_{0.4}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{X}/R_{0.4} = \{ (a,d), \{b\}, \{c\}\} \Leftarrow \alpha = 0.4$$

$\alpha = 1$ gives identity

$$\underline{X}/R_1 = \{ \{a\}, \{b\}, \{c\}, \{d\}\}$$



Partition
Tree

3.5 Compatibility or Tolerance Relations

crisp R { reflexive, symmetric } → R is a 'compatibility' relation or 'tolerance' relation

Fuzzification ⇒ 'proximity relation'

Crisp R ⇒ compatibility class, $A \overset{\text{subset}}{\subset} X \ni xRy \; \forall (x,y) \in A$

$xRy \rightarrow (x,y) \in R$

Example:

$$\begin{array}{cccc} a & b & c & d \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

{a,b}
{a,d}
{b,d}

$\{\{a,b,d\},$ ← plus, all subsets {a} {b}, {c}, {d} $, \{b,c\}\}$ = set of all compatibility classes

$$\begin{array}{c|ccc} & a & b & d \\ \hline a & 1 & 1 & 1 \\ b & 1 & 1 & 1 \\ d & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} & b & c \\ \hline b & 1 & 1 \\ c & 1 & 1 \end{array}$$

a maximal compatibility class is not contained in any other compatibility class
~~⇒ no member is a sub set of of another member~~

complete cover

= set of al maximal compatibility classes.

= { {a, b, d}, {b, c} }

Proximity Relation

= Fuzzy compatibility relation

$\Rightarrow$ reflexive & symmetric

α cuts are like 'compatibility classes'

'Maximal α-compatibles'
~~α~~· are not subsets of
'α-compatible classes'

Example:

$$M_R = \begin{bmatrix} 1 & 0.1 & 0 & .3 \\ 0.1 & 1 & 0 & 0 \\ 0 & 0 & 1 & .2 \\ .3 & 0 & .2 & 1 \end{bmatrix}$$

$$M_{R_{0.1}} = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \Longrightarrow \begin{array}{l} \alpha = 0.1 \\ \text{maximal} \end{array} = \{ \{a, b, d\}, \{c, d\} \}$$

$\alpha = 0.1$
compatible
class

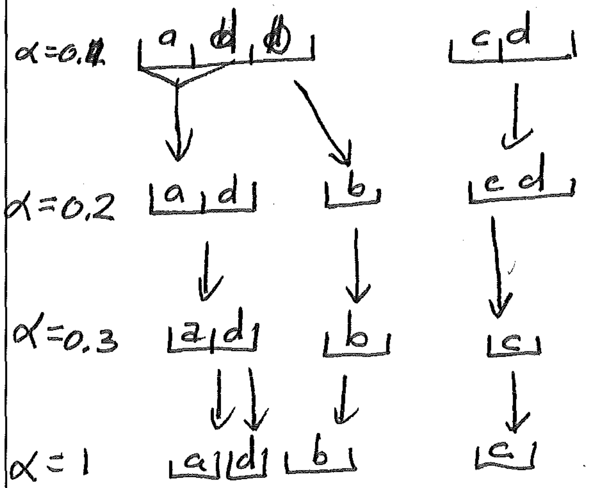
At a level $\alpha = 0.1$, a, b & c are compatible

$$M_{R_{0.2}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad \alpha = 0.2 \Rightarrow \{ (a,d\}, \{b\}, \{c,d\})$$

$$M_{R_{0.3}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \alpha = 0.3 \Rightarrow \{ (a,d\}, \{b\}, \{c\}$$

$$M_{R_1} = I \qquad \{a, b, c, d\}$$

Complete α cover ~~graph~~ tree



3.6. ORDERINGS

CRISP: Reflexive, Antisymmetric, Transitive $\}$ A Partial Ordering

Antisymmetric:

$(x,y) \in R,\ (y,x) \in R \Rightarrow x = y$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Leftarrow \text{Antisymmetric}$$

Definitions:

$x \leq y \Rightarrow (x,y) \in R$, 'x preceeds y'

$x \leq y,\ \nexists\ z \ni x \leq z,\ z \leq y$

$\Rightarrow$ x is 'immediate predesessor of y'

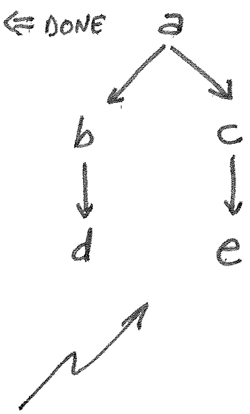
Example

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline \hat{a} & 1 & 0 & 0 & 0 & 0 \\ \hat{b} & \underline{1} & 1 & 0 & 0 & 0 \\ \hat{c} & \underline{1} & 0 & 1 & 0 & 0 \\ \hat{d} & 1 & \underline{1} & 0 & 1 & 0 \\ \hat{e} & 1 & 0 & \underline{1} & 0 & 1 \end{array}$$

partial ordering:

$a, \hat{b}$; $b, \hat{d}$ ⇐ DONE

$a, \hat{c}$; $c, \hat{e}$



Hasse diagram

immediate predecessor & successor

Can't use $a\hat{d} \xRightarrow[\text{cause}]{} (a\ b)\ (b, z)$

Definitions

$x \in \mathbb{X},\ x \leq y\ \forall\ y \in \mathbb{X} \Rightarrow x$ is 'first member of $\mathbb{X}$' } may exist

$x \in \mathbb{X},\ y \leq x\ \forall\ y \in \mathbb{X} \Rightarrow x$ is 'last " " "' }

$x \in \mathbb{X},\ y \leq x \Rightarrow x = y \implies x$ is 'a minimal " " "' } Always Exist

$x \in \mathbb{X},\ x \leq y \Rightarrow x = y \Rightarrow x$ is 'a maximal " " "' }

if $\exists$ first member $\Rightarrow$ first member $= !\ \underbrace{\text{minimal member}}_{\text{unique}}$

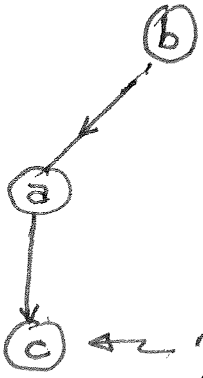
if $\exists$ last " $\Rightarrow$ last member $= !$ maximal member

Example

$$\begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 1 & 0 & 0 \\ \hat{b} & 0 & 1 & 0 & 0 \\ \hat{c} & 1 & 1 & 1 & 0 \\ \hat{d} & 0 & 0 & 0 & 1 \end{array}$$

$a \le \hat{c}$

$\underset{\frown}{b} \le a \qquad b \le c$



?

~~maxim~~al element

$b \le y \Rightarrow \quad y = b$

?

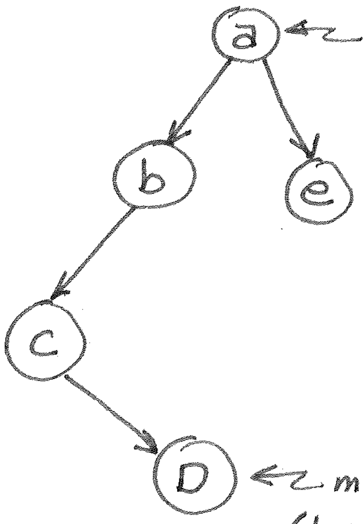
mi~~ni~~mal element

$x \le c \Rightarrow x = c$

d is "noncomparable"

Example

$$\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \underline{1} & 1 & 0 & 0 & 0 \\ 1 & \underline{1} & 1 & 0 & 0 \\ 1 & 1 & \underline{1} & 1 & 0 \\ \underline{1} & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$



maximal & last $x \leq a \ \forall x \in X$

Hasse Diagram

minimal (but not first: not less than e)

UPPER BOUND:

Let $A \subseteq \mathbb{X}$ (subset)

* $x \in \mathbb{X}$, $x \leq y \; \forall \; y \in A \Rightarrow x$ is a lower bound

* $x \in \mathbb{X}$, $y \leq x \; \forall \; y \in A \Rightarrow x$ is an upper bound

* $x =$ lower bound, $x \leq z \; \forall \; z =$ lower bound

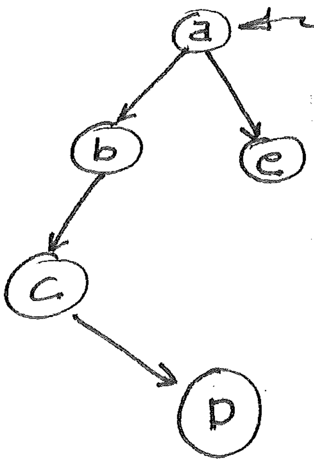
$\Rightarrow x$ is 'greatest lower bound' or 'infinum'.

* $x =$ upper bound, $z \leq x \; \forall \; z =$ upper bound

$\Rightarrow x$ is 'least upper bound' or 'supremum'

If an ordering contains a LUB & GLB $\forall \; A \subseteq \mathbb{X}$, it is a lattice.

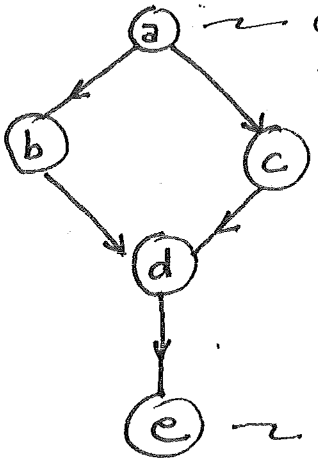
Previous Example:



No Least upper Bound.

$\nexists$ Upper Bound for
$A = \{b, e\}$

Example



Q: Is Last Element = GLB ?

Is First " = LUB ?

fuzzy partial ordering } → reflexive
→ antisymmetric
→ transitive

fuzzy antisymmetric:

$$\mu_R(x,y) > 0 \ \& \ \mu_R(y,x) > 0 \Rightarrow x = y$$
$$\forall (x,y) \in R$$

Example of antisymmetric

$$\begin{matrix} 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .8 \\ 0 & .2 & 1 & 0 \\ .4 & 0 & 0 & 1 \end{matrix} \quad \Leftarrow \quad \text{(non reflexive non transitive)}$$

Example of Partial fuzzy ordering:

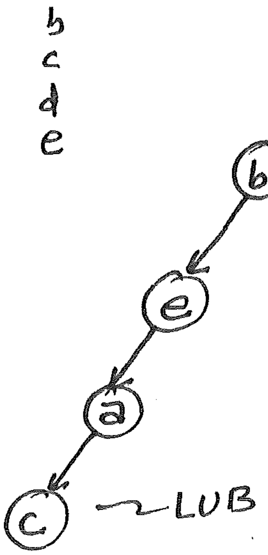
$$M_R = \begin{array}{c|ccccc} & a & b & c & d & e \\ a & 1 & 0.7 & 0 & 1 & .7 \\ b & 0 & 1 & 0 & .9 & 0 \\ c & .5 & .7 & 1 & 1 & .8 \\ d & 0 & 0 & 0 & 1 & 0 \\ e & 0 & .1 & 0 & .9 & 1 \end{array}$$

Each α cut of a fuzzy partial ordering is a crisp partial ordering

★ $M_{R_{0.1}} = \begin{array}{ccccc|c} 1 & 1 & 0 & 1 & \underline{1} & a \\ 0 & 1 & 0 & \underline{1} & 0 & b \\ \underline{1} & 1 & 1 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & d \\ 0 & 1 & 0 & 1 & 1 & e \\ a & b & c & d & e & \end{array}$

Note:

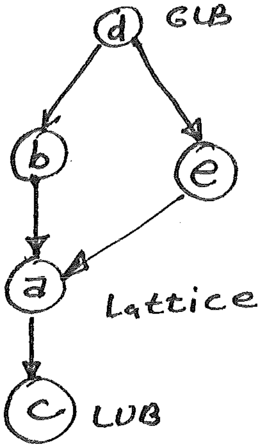
$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0. \\ d & b & e & a & e \end{array}$



complete A^{v} linear ordering (always a lattice)

★ $M_{R_{0.5}} = \begin{array}{ccccc} a & b & c & d & e \\ 1 & \underline{1} & 0 & 1 & \underline{1} \\ 0 & 1 & 0 & \underline{1} & 0 \\ \underline{1} & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \textcircled{0} & 0 & \underline{1} & 1 \end{array}$

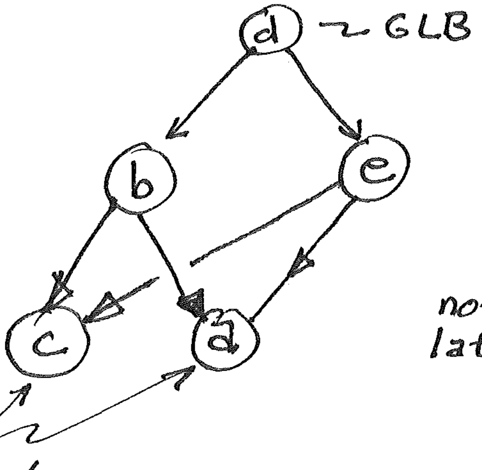
only change



Lattice

★ $M_{R_{0.7}} = \begin{array}{ccccc} a & b & c & d & e \\ 1 & \underline{1} & 0 & 1 & \underline{1} \\ 0 & 1 & 0 & \underline{1} & 0 \\ \textcircled{0} & \underline{1} & 1 & 1 & \underline{1} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \underline{1} & 1 \end{array}$

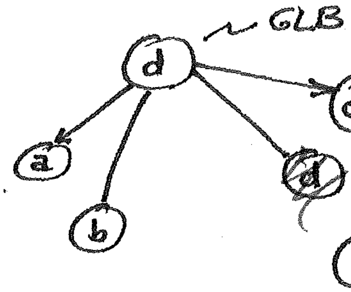
Change



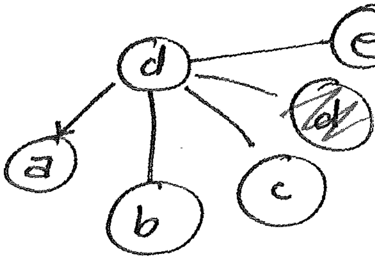
not a lattice

minimal elements

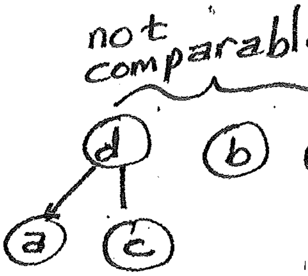
$$M_{R_{0.8}} = \begin{array}{ccccc} a & b & c & d & e \\ 1 & \textcircled{0} & 0 & 1 & \textcircled{0} \\ 0 & 1 & 0 & 1 & 0 \\ 0 & \textcircled{0} & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}$$



$$M_{R_{0.9}} = \begin{array}{ccccc} a & b & c & d & e \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & \textcircled{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}$$



$$M_{R_1} = I = \begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \textcircled{0} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \textcircled{0} & 1 \end{array}$$



Fuzzy Partial Orderings give rise to two Fuzzy sets $\forall\, x \in \mathbf{X}$.

Dominating: The degree to which other elements precede

Dominated: The degree to which other elements succeed

★ DOMINATING CLASS

$R_{\geq[x]}$ = Degree of DOMINANCE OVER x

= members of $\mathbf{X}$ to the degree to which they dominate x

$\mu_{R\geq[x]}(y) = \mu_R(x, y)$

the degree in which each element precedes x

Example:

$$M_R = \begin{array}{c|ccccc|c} x \downarrow \; y \rightarrow & a & b & c & d & e & \\ & 1 & .7 & 0 & 1 & .7 & a \\ & 0 & 1 & 0 & .9 & 0 & b \\ & .5 & .7 & 1 & 1 & .8 & c \\ & 0 & 0 & 0 & 1 & 0 & d \\ & 0 & .1 & 0 & .9 & 1 & e \end{array}$$

(Rows

$R_{\geq[c]} = 0.5/a + 0.7/b + 1/c + 1/d + 0.8/e$

$R_{\geq[d]} = 1/d \Leftarrow$ The element d is 'undominated'

nothing proceeds

★ DOMINATED

$R_{\leq[x]}$ = Degree of DOMINANCE OF x

= members of X to the degree which they are dominated by x

$$\mu_{R\leq[x]}(y) = \mu_R(y,x)$$

degree to which each element succeeds x

Using relation matrix on previous page:

$R\leq[e] = 0.7/a + 0.8/c + 1/e$

$R\leq[c] = 1/c \Leftarrow$ c is 'undominating'

nothing succeeds

Given $A \subseteq X$, the fuzzy upper bound for A :

$$U(R,A) = \bigcap_{x \in A} R_{\geq[x]}$$

Example: From matrix, let $A = \{a, b\}$

$$R_{\geq[a]} = 1/a + 0.7/c + 1/d + 0.7/e$$

$$R_{\geq[b]} = 1/b + .9/d$$

$$\bigcap_{x \in A} R_{\geq[x]} = 0.7/b + 0.9/d = U(R, \{a,b\})$$

(FLUB!)

fuzzy least upper bound: (of set A)

(<u>if</u> it exists)

$= ! \; x \in U(R,A) \ni$

unique

$$\mu_{U(R,A)}(x) > 0 \quad \underline{\text{and}} \quad \mu_R(x,y) > 0 \; \forall \; y \text{ in range of } U$$

FOR OUR EXAMPLE

$$\mu_{U(R,A)}(x) > 0 \Rightarrow x = b \text{ or } d$$

From p. 110 $\begin{cases} \mu_R(b,d) = 0.9 \\ \mu_R(d,b) = 0 \end{cases}$

$\Rightarrow$ F.L.U.B. = b

EX

$A = \{a, c\}$

$U(R;A) = 0.5/a + 0.7/b + 1/d + 0.7/e$

$$\mu_R(x,y)$$

$$\begin{array}{cccccc|l} & & a & b & d & e & \longrightarrow y \\ & a & 1 & 0.7 & 1 & 0.7 & \rightarrow \text{all} > 0 \\ \cline{3-6} x \downarrow & b & 0 & 1 & .9 & 0 & \\ & d & 0 & 0 & 1 & .8 & \\ & e & 0 & .1 & .9 & 1 & \end{array}$$

$\Rightarrow$ FLUB = a

EX

$A = \underline{X}$

$U(R; \underline{X}) = 0.9 d$

Clearly, FLUB = 0.9 d

HW: Find FGLB

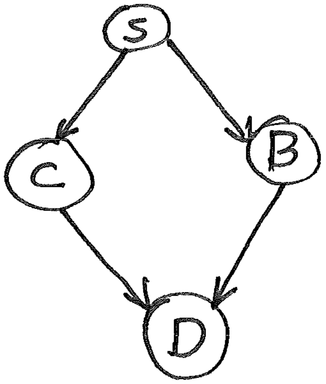
Example:

$\mathbb{X} = \{$smoking, cancer, bed burning, death$\}$

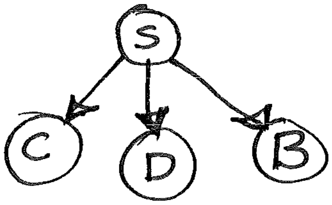
$$M_R = \begin{array}{c|cccc} & \multicolumn{4}{l}{\xrightarrow{\quad} y} \\ x\downarrow & S & C & B & D \\ S & 1 & 0 & 0 & 0 \\ C & 0.9 & 1 & 0 & 0 \\ B & 0.8 & 0 & 1 & 0 \\ D & 1 & 0.7 & 0.7 & 1 \end{array}$$

α cuts

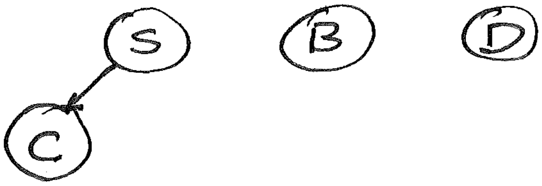
$$M_{R_{0.7}} = \begin{array}{c|cccc} & S & C & B & D \\ S & 1 & 0 & 0 & 0 \\ C & 1 & 1 & 0 & 0 \\ B & 1 & 0 & 1 & 0 \\ D & 1 & 1 & 1 & 1 \end{array}$$



$$M_{R_{0.8}} = \begin{array}{cccc} S & C & B & D \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & \textcircled{0} & \textcircled{0} & 1 \end{array}$$



$$M_{R_{0.9}} = \begin{array}{c} \begin{array}{cccc} S & C & B & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



M_{R_1} ⇐ α cut used by tobacco lobby

Fuzzy Least Upper Bound

$$U(R; \underline{X}) = 0.8/S$$

S is FLUB

$S = \{D, B\}$

$$U(R; S) = 0.8/S + 0.7/B$$

$$\mu_R(S, B) = 0$$

$$\mu_R(B, S) = 0.8$$

$\Rightarrow$ B is FLUB

Makes sense!

★ Dominating Class:

$R_{\geq[D]} = 1/S + 0.7/C + 0.7/B + 1/D$

= degree of dominance over death by smoking, cancer, bed burning & death

$R_{\geq[B]} = 0.8/S + 1/B$

★ Dominated Class

$R_{\leq[S]} = 1/S + 0.9/C + 0.8/B + 1/D$

= degree that smoking is dominated by smoking, cancer, bed burning & death

$R_{\leq[C]} = 1/C + 0.7/D$

3.7. MORPHISMS

★ CRISP:

Given $R(\mathbf{X},\mathbf{X})$ & $Q(\mathbf{Y},\mathbf{Y})$
then function

$$h: \mathbf{X} \rightarrow \mathbf{Y}$$

is a 'homomorphism'

from $(\mathbf{X}, R)$ to $(\mathbf{Y}, Q)$ iff

$$(x_1, x_2) \in R \Longrightarrow (h(x_1), h(x_2)) \in Q$$

$\forall\ (x_1\ x_2) \in \mathbf{X}$

ie $(x_1$ & $x_2)$ related by R
$\Rightarrow (h(x_1)$ & $h(x_2))$ related by Q

★ FUZZY

R & Q fuzzy

$$\mu_R(x_1, x_2) \leq \mu_Q(h(x_1), h(x_2))$$

$\forall\ (x_1, x_2) \in \mathbf{X}$

Thus, if $(x_1$ & $x_2)$ are fuzzily related by R, $(h(x_1)$ & $h(x_2))$ are more strongly related by Q.

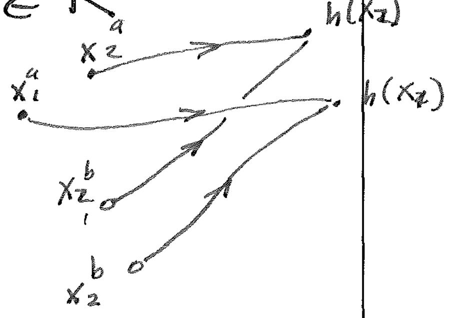
★ STRONG CRISP HOMOMORPHISM

$(x_1, x_2) \in R \Rightarrow (h(x_1), h(x_2)) \in Q$

$\forall \; x_1, x_2 \in \underline{X}$ and

$(y_1, y_2) \in Q \Rightarrow (x_1, x_2) \in R$

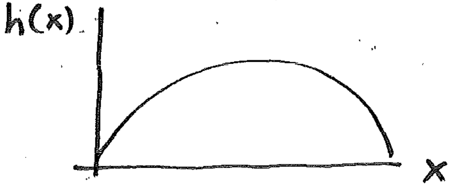
$\forall \; y_1, y_2 \in \underline{Y}$



Note: h is one '

h need not be one to one

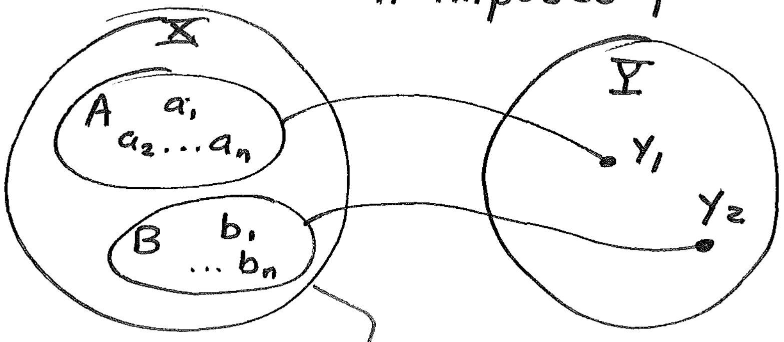
~~h^{-1}~~ h can be many to one:



$\Rightarrow h^{-1}(y)$ is a set of solutions.

★ FUZZY STRONG HOMOMORPHISMS

h imposes partition on $\mathbb{X}$



$y_1 = h(a_n) \in \mathbb{Y}$ Block Partition, π_h

$y_2 = h(b_n) \in \mathbb{Y}$

For R on $\mathbb{X}$

$\ddagger$ Q on $\mathbb{Y}$

STRONG HOMOMORPHISM FROM $\{\mathbb{X}, R\}$ to $\{\mathbb{Y}, Q\}$ iff, $\forall$ blocks of the partition

$$\max_{i\,j} \mu_R(a_i, b_j) = \mu_Q(y_1, y_2)$$

Example 3.20

$\mathbb{X} = \{a, b, c, d\}$ $\qquad$ $\mathbb{Y} = \{\alpha, \beta, \gamma\}$

$$M_R = \begin{array}{c|cccc} x_1 \downarrow \; \overrightarrow{x_2} & a & b & c & d \\ & 0 & .5 & 0 & 0 \\ & 0 & 0 & .9 & 0 \\ & 1 & 0 & 0 & .5 \\ & 0 & .6 & 0 & 0 \end{array} \qquad M_Q = \begin{array}{c|ccc} y_1 \downarrow \; \overrightarrow{y_2} & \alpha & \beta & \gamma \\ & .5 & .9 & 0 \\ & 1 & 0 & .9 \\ & 1 & .9 & 0 \end{array}$$

Homomorphic Map

$\alpha = h(a) = h(b)$

$\beta = h(c)$

$\gamma = h(d)$

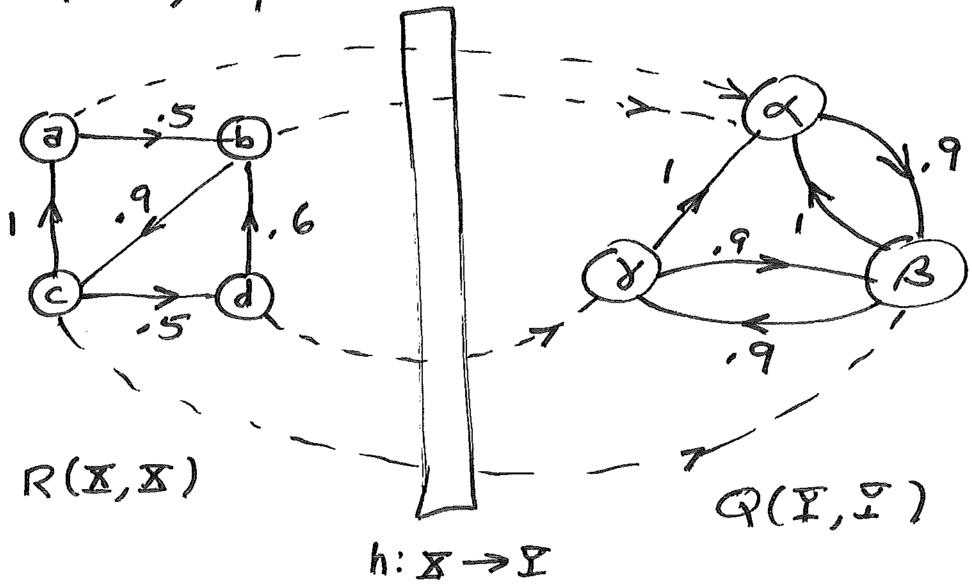
$\geq \mu_R(x_1, x_2)$

9 Blocks in π_h

$(x_1\ x_2)$	$\mu_R(x_1, x_2)$	(y_1, y_2)	$\mu_Q(y_1, y_2)$	
$(a\ a)$	0	(α, α)	.5	
(a, b)	.5	"	"	✓
(b, a)	0	"	"	
(b, b)	0	"	"	
(a, c)	0	(α, β)	.9	✓
(b, c)	.9	"	"	
(a, d)	0	(α, γ)	0	✓
(b, d)	0	"	"	
(c, c)	0	$(\beta\ \beta)$	0	✓
(c, d)	0.5	(β, γ)	$\boxed{0.9}$ $\rightarrow$	
(d, a)	0	(γ, α)	$\boxed{1}$ $\rightarrow$	
(d, b)	0.6	"		
(d, c)	0	(γ, β)	$\boxed{0.9}$ $\rightarrow$	
(d, d)	0	(γ, γ)	0	✓
(c, a)	1	(β, α)	1	✓
(c, b)	1	"	"	✓

Sagittal Oiagram

p.93, top



This becomes strong if

$\mu_R(\beta, \gamma) = 0.5$ (from 0.9)

$\mu_R(\gamma, \alpha) = 0.6$ (from 1)

$\mu_R(\gamma, \beta) = 0$ (from 0.9)

Homomorphism $(\mathbf{X}, R) \rightarrow (\mathbf{Y}, Q)$

If h is completely specified and one-to-one and onto

$\Rightarrow$ Isomorphism

3.8 FUZZY RELATION EQUATIONS

Consider Composition

$$R = P \circ Q$$

or

$$r_{ik} = \max_{j} \min(p_{i,j}, q_{jk})$$

FUZZY RELATION EQUATIONS

Given P & Q, find R ⇐ easy

Given Q & R, find P ⇐ <u>much</u> harder

Indeed, solution is generally not unique.

Define solutions set:

$$S(Q,R) = \{ P \mid P \circ Q = R \}$$

$$\begin{bmatrix} \quad P \quad \end{bmatrix} \circ \begin{bmatrix} \quad Q \quad \end{bmatrix} = \begin{bmatrix} \quad R \quad \end{bmatrix}$$

Can do a row at a time

$$P = \begin{bmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \vdots \\ \vec{p}_n \end{bmatrix} ; \quad R = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{bmatrix}$$

$$p_i \circ Q = r_i$$

Let simplified solution set be denoted by

$$S_i(Q, r_i) = \{ p_i \mid p_i \circ Q = r_i \}$$

Can we note if a solution is not possible?

The p_{ij}'s must satisfy:

$$\max[\min(p_{i1}, q_{1k}), \quad \min(p_{i2}, q_{2k}) \quad \ldots \min(p_{in}, q_{ik})] = r_{ik}$$

$$\max_j \min(p_{ij}, q_{jk}) = r_{ik}$$

For a given k, the biggest q_{jk} better $\geq r_{ik}$. That is

$$\max_j q_{jk} \geq r_{ik}$$

If $\max_j q_{jk} < r_{ik} \Leftarrow$ no solution

Example:

$$[p_{i1} \quad p_{i2} \quad p_{i3}] \circ \begin{bmatrix} 0.2 \\ 0.5 \\ 0.7 \end{bmatrix} = 0.8$$

There is no solution. The maximum we can get is 0.7

★ An inequality:

$$r_{ik} = \max_j \min[p_{ij}, q_{jk}]$$

$$\leq \max_j q_{jk}$$

No solution if

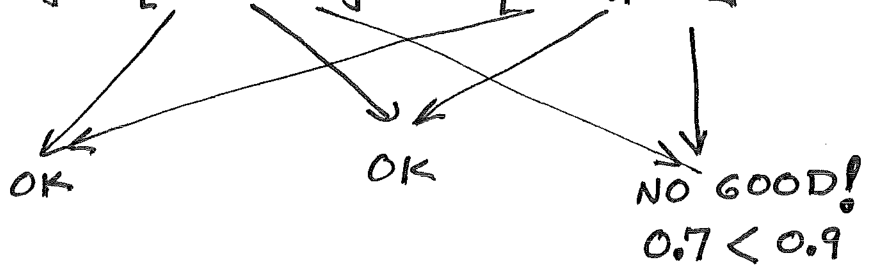
$$\max_j q_{jk} < r_{ik}$$

This must hold $\forall\, i$. Generalization

if $\exists k \ni \max_j q_{jk} < \max_i r_{ik}$ $\Rightarrow$ no solution

Example:

$$i\downarrow \overset{j\rightarrow}{\begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix}} \circ \; j\downarrow \overset{k\rightarrow}{\begin{bmatrix} 0 & 1 & 0.7 \\ .1 & 0 & 0.1 \\ 1 & .4 & 0.3 \end{bmatrix}} = i\downarrow \overset{k\rightarrow}{\begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 0 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}}$$



No solution here.

If there is no solution,

$$S(Q,R) = \phi$$

Consider when a solution does exist:

$$S(Q,r) = \{p \mid p \circ Q = r\} \neq \phi$$

Notation:

$^1p \le {}^2p$ if each element in p^1 is $\le$ " corresponding element in p^2.

ie $\quad {}^1p_j \le {}^2p_j \quad \forall j \in J$

Define

$$\langle {}^1p, {}^2p \rangle = \{p \mid {}^1p \le p \le {}^2p\}$$

If $S(Q,r) \neq \phi$, $\exists!$ maximal $p = \hat{p} \in S$

$\exists$(possibly numerous) minimal solutions, $p = \check{p} \in S$

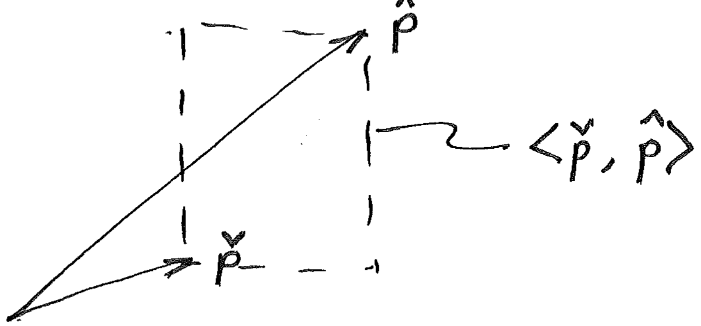
$$\check{p} \in S \;, \hat{p} \in S \Rightarrow \langle \check{p}, \hat{p} \rangle \in S$$

Since there can be numerous minimal solutions

crisp union

$$S(Q,r) = \bigcup_{\hat{p}} \langle \check{p}, \hat{p} \rangle$$

In 2-D



Example:

$$Q = \begin{array}{cc} & \xrightarrow{k} \\ j \downarrow & \begin{bmatrix} .1 & .4 & .5 & .1 \\ .9 & .7 & .2 & 0 \\ .8 & 1 & .5 & 0 \\ .1 & .3 & .6 & 0 \end{bmatrix} \end{array} \leftarrow j=1 \qquad r = \overset{\xrightarrow{k}}{[}\underset{\uparrow}{0.8} \;\; .7 \;\; .5 \;\; 0]$$

Find $\hat{p}$:

$$\hat{p}_1 = \min(1, 1, 1, 0) = 0$$
$$\hat{p}_2 = \min(0.8, 1, 1, 1) = 0.8$$
$$\hat{p}_3 = \min(1, .7, 1, 1) = 0.7$$
$$\hat{p}_4 = \min(1, 1, .5, 1) = 0.5$$

$$\hat{p} = [0, .8, .7, .5]$$

Finding minimal solutions is a bear!
Algorithm on p. 99 of text.

Finding solution for:

$$[\quad p_j \quad] \circ \overset{\xrightarrow{k}}{\begin{bmatrix} \\ q_{jk} \\ \\ \end{bmatrix}} = [\;\overset{\xrightarrow{k}}{r_k}\;]$$

maximum solution for p:

$$\hat{p}_j = \min_k \sigma(q_{j,k}, r_k)$$

$$\sigma(q_{jk}, r_k) = \begin{cases} r_k & ; q_{jk} \overset{\text{STRICT}}{>} r_k \\ 1 & ; \text{else} \end{cases}$$

Fuzzy If-Then Rules

EX: FUZZY WASHING MACHINE

★ Parameters : Linguistic Variables

WL = Wash Load
CQ = Cloth Quality } Input
WT = Washing Time ← Output

} Each has universe of discourse

★ Linguistic Variables:

WL → L, A, H (Light, Average, Heavy)
CQ → S, $\bar{S}$ (Soft)
WT → VS, S, A, L (Very Short, Short, Medium, Long)

★ Rules (Fuzzy Implications)

If WL is L and CQ is S then WT is VS
or " " " L " " " $\bar{S}$ " " " S
" " " A " " " S " " " S
" " " A " " " $\bar{S}$ " " " A
" " " H " " " S " " " A
" " " H " " " $\bar{S}$ " " " L

antecedents — consequents

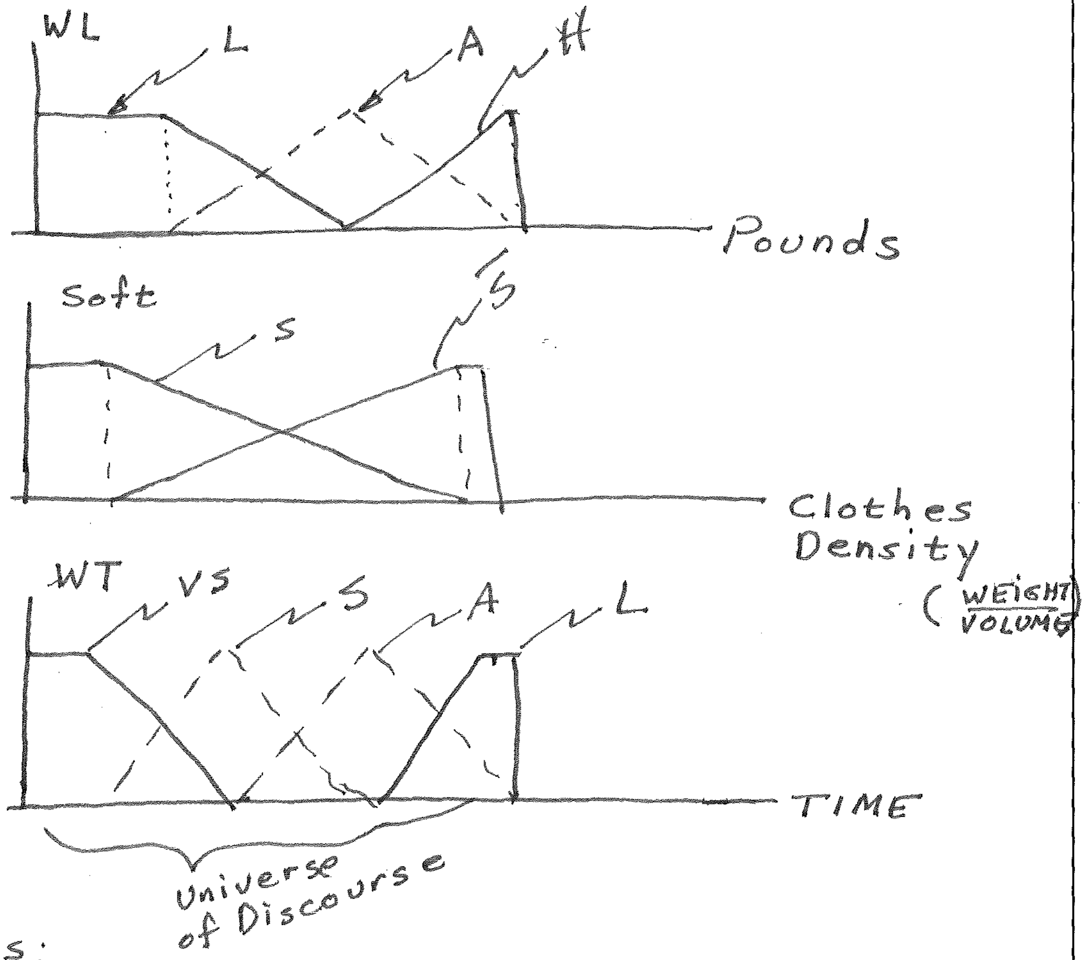
Fuzzy Rule Table

		WL	
	L	A	H
S	VS	S	A
$\bar{S}$	S	A	L

Alternate Statement of Rules

If [WL is L & CQ is S] then WT 1
OR If [WL " L " " " $\bar{S}$
OR " " " A " " " S] then " 2
OR If [WL " A " " " $\bar{S}$
OR " " " H " " " S] " "
OR If [WL " H " " " $\bar{S}$] " "

We must decide what we mean by linguistic variables:



Notes:

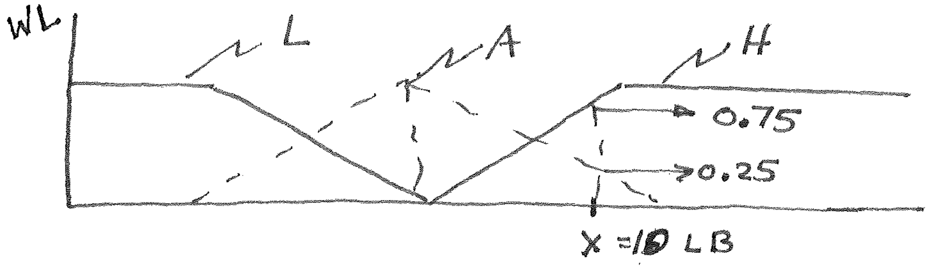
1. We are performing 'sensor fusion'
2. Fuzzy memberships set by
   - expert
   - trial & error
   - adaption (more recently)

How it works:

1. Measure inputs (WL, S)

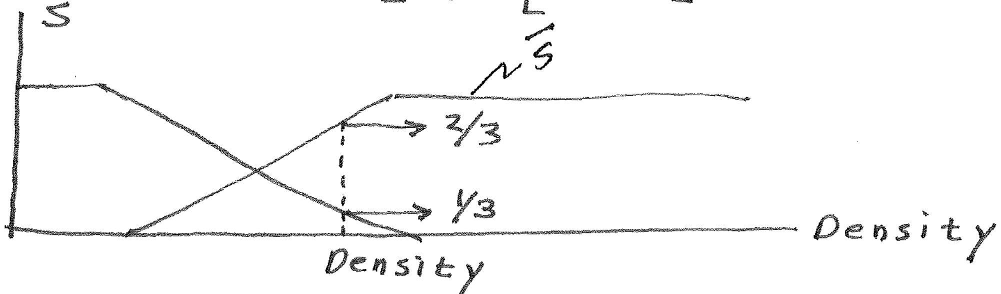
2. 'Fuzzify'

Place input into a vector of membership values



X = 10 LBS GIVES MEMBERSHIP VECTOR

$$\begin{bmatrix} L \\ A \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 0.75 \end{bmatrix}$$



$$\begin{bmatrix} S \\ \bar{S} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

1. WL is L & CQ is S

$= \min\left(0, \frac{1}{3}\right) = 0$

2. WL is L & CQ is $\bar{S}$ <u>OR</u> WL is A & CQ is S

$= \max\left[\min\left(0, 2/3\right), \min\left(1/4, 1/3\right)\right] = 1/4$

3. WL is A & CQ is $\bar{S}$ <u>OR</u> WL is H & CQ is S

$= \max\left[\min\left(1/4, 2/3\right), \min\left(3/4, 1/3\right)\right] = 1/3$

4. WL is H & CQ is $\bar{S}$

$= \min\left(0.75, 2/3\right) = 2/3$

Rules:

If [0] then WT is VS
If [1/4] " " " S
If [1/3] " " " A
If [2/3] " " " L

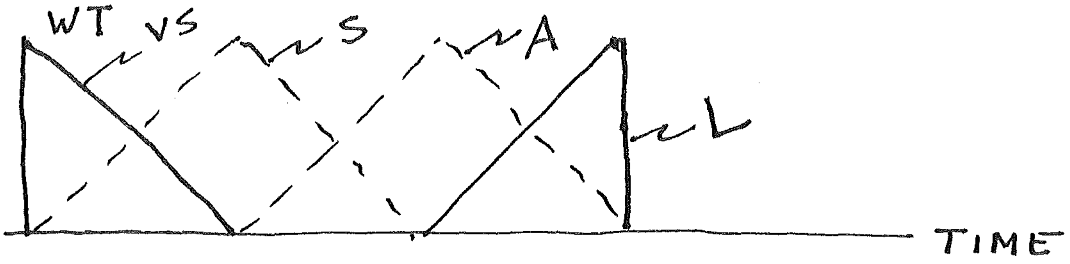
Question:

How do we transform these fuzzy degrees of truth into a crisp decision

Answer:

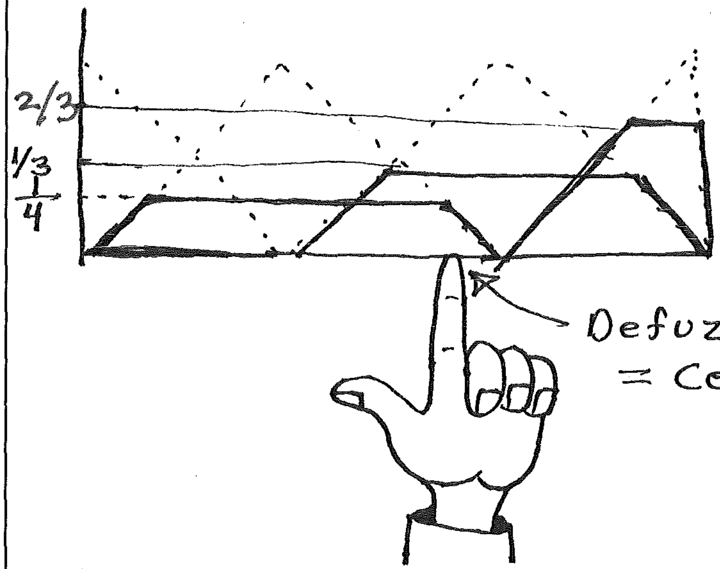
<u>De</u>fuzzify

To do so, we must specify what we mean by



One method of defuzzification:

1. Cut output consequences by the possibility of the corresponding antecedents



Defuzzified Time t_0 = Center of Mass

$\mu_n^c(t)$ = 'cut' membership

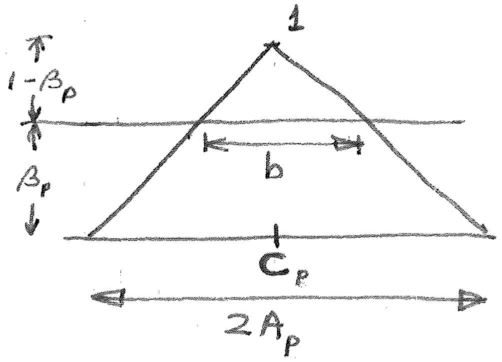
$$t_0 = \sum_n \int t \mu_n^c(t)dt \Big/ \sum_n \int \mu_n^c(t)dt$$

$$= \sum A_n^c c_n \Big/ \sum A^c$$

centroid

Minimum Cut for triangles

A_p = AREA OF pTH TRIANGLE
β_p = cut



$$\frac{b}{1-\beta} = \frac{2A}{1} \Rightarrow b = 2(1-\beta)A$$

$$\text{AREA OF TOP TRIANGLE} = \tfrac{1}{2} b(1-\beta)$$
$$= (1-\beta)^2 A$$

$$\text{AREA OF TRAAPEZOID} = [1-(1-\beta)^2]A$$
$$= [1 - 1 + 2\beta - \beta^2]A \doteq \beta(2-\beta)A$$

Thus

$$\bar{z} = \frac{\sum_p \int z\, \mu_p^{\beta_p}(z)\, dz}{\sum_p \int \mu_p^{\beta_p}(z)\, dz}$$

$$\int \frac{z\,\mu_p^{\beta}(z)}{\int \mu_p^{\beta}(z)\,dz} = C \Rightarrow \int z\,\mu_p^{\beta}(z)\,dz = A_p C$$
$$= \beta(\beta+2)AC$$

Thus

$$\bar{z} = \frac{\sum_p \beta_p(\beta_p - 2)A_p C_p}{\sum_p \beta_p(\beta_p - 2)A_p} \longleftarrow \text{only for triangles}$$

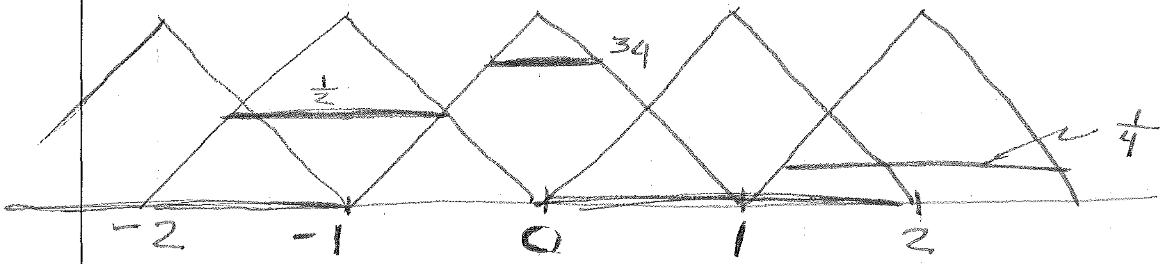
Special case

$A_p = A \ \forall \ p$

$C_p = \Delta p$

$$\Rightarrow \bar{z} = \frac{\Delta \sum_p p\beta_p(2-\beta_p)}{\sum_p \beta_p(2-\beta_p)} \longleftarrow \text{TRIANGLES ONLY}$$

Ex: $\Delta = 1, \ A_p = p$



$$\bar{z} = \frac{-2\cdot0\cdot2 + -1\,\frac{2}{4}\frac{6}{4} + 0\,\frac{3}{4}\cdot\frac{5}{4} + 1\cdot0\cdot2 + 2\,\frac{1}{4}\cdot\frac{7}{4}}{0\cdot2 + \frac{2}{4}\cdot\frac{6}{4} + \frac{3}{4}\cdot\frac{5}{4} + 0\cdot2 + \frac{1}{4}\cdot\frac{7}{4}}$$

$$= \frac{-12+14}{12+15+7} = \frac{2}{34} = \frac{1}{17} \approx 0$$

Summary.

1. Fuzzify the inputs, x, y

$$x \longrightarrow \begin{bmatrix} \mu_{A_1}(x) \\ \mu_{A_2}(x) \\ \vdots \\ \mu_{A_n}(x) \end{bmatrix} = \vec{\mu}_A$$

$$y \longrightarrow \begin{bmatrix} \mu_{B_1}(y) \\ \mu_{B_2}(y) \\ \vdots \\ \mu_{B_m}(y) \end{bmatrix} = \vec{\mu}_B$$

2. Evaluate antecedents of fuzzy if-then rules:

$$\text{If } \underbrace{A_n \text{ and } B_m}_{\text{Antecedent}} \text{ then } C_p$$

This corresponds to a cartesian product:

$$\cancel{\beta_{nm}} = \min \underbrace{\mu_{A_n}(x), \mu_{B_n}(y)}_{\text{elements of } \vec{\mu}_A \circ \vec{\mu}_B}) = \beta_P$$

COMPOSITION

3, Defuzzify

Cut membership functions, $\mu_{C_p}(z)$, by β_p

Call them $\mu_{C_p}^{\beta_p}(z)$

Find corresponding center of mass:

★ General:

$$\bar{z} = \frac{\sum_p \int z\, \mu_{C_p}^{\beta_p}(z)\,dz}{\sum_p \int \mu_{C_p}^{\beta_p}(z)\,dz}$$

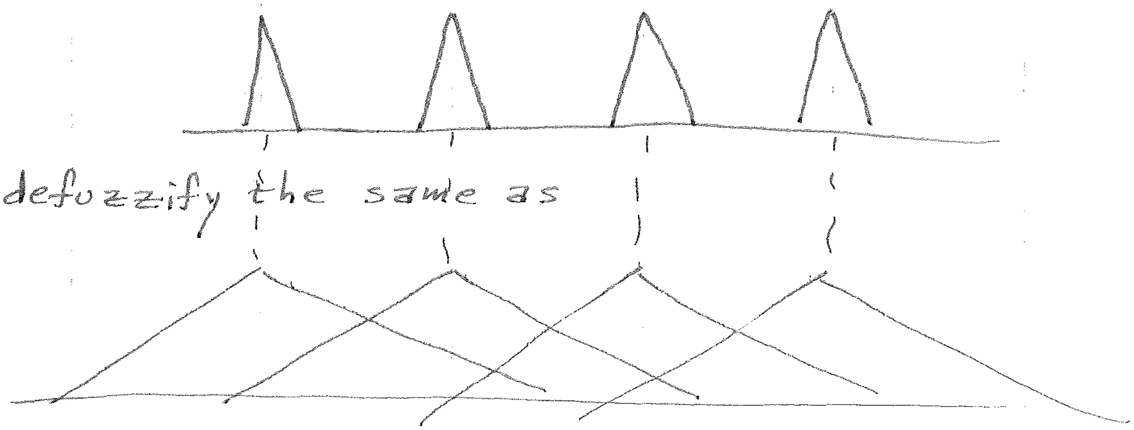
★ For ~~isocoles~~ triangles: A_p=area, C_p=centroid

$$\bar{z} = \frac{\sum_p \beta_p(2-\beta_p)\, C_p A_p}{\sum_p \beta_p(2-\beta_p) A_p}$$

★ For identical equally spaced triangles:

$$\bar{z} = \Delta \frac{\sum_p p\, \beta_p(2-\beta_p)}{\sum_p \beta_p(2-\beta_p)}$$

Does this:



defuzzify the same as

?

- VARIATIONS

(a) Evaluate antecedent possibility using product

$$\beta_{nm} = \mu_{A_n}(x)\,\mu_{B_n}(y) = \beta_P$$

$$\Rightarrow \vec{\mu}_A\,\vec{\mu}_B^{\,T}$$

instead of

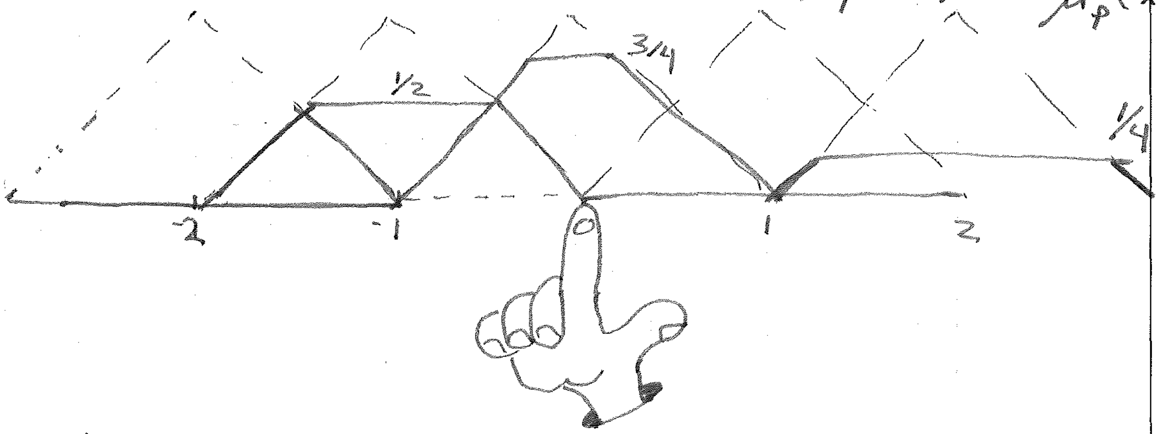
$$\beta_{nm} = \min[\mu_{A_n}(x), \mu_{B_m}(y)]$$

$$\Rightarrow \vec{\mu}_A \circ \vec{\mu}_B$$

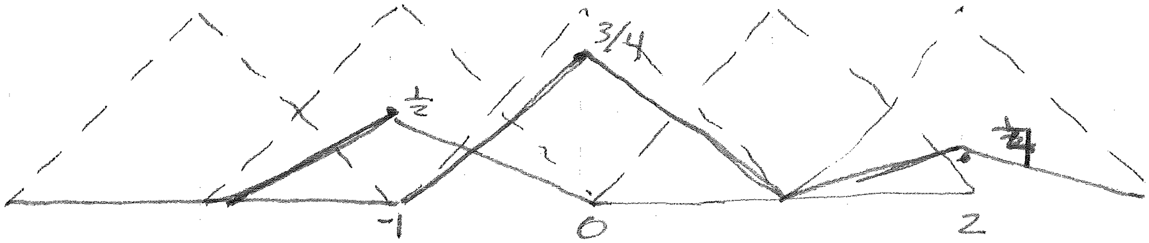
(b) USING WEIGHTS INSTEAD OF CUTS IN DEFUZZIFICATION

cut

$$\mu_p(x) \longrightarrow \mu_p^{\beta_p}(x) = \begin{cases} \mu_p(x) ; & \mu_p(x) < \beta_p \\ \beta_p ; & \mu_p(x) > \beta_p \end{cases}$$



Weight



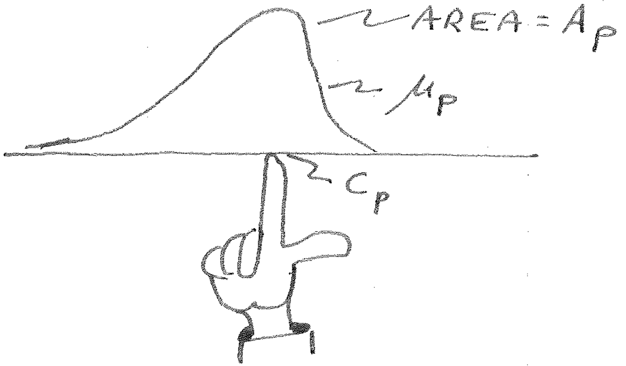
$$\mu_p(x) \longrightarrow \beta_p \mu_p(x)$$

$$\Rightarrow \bar{z} = \frac{\int \sum_p z \beta_p \mu_p(z)}{\int \sum_p \beta_p \mu_p(z)}$$

Let
$$\frac{\int z \mu_p(z)\,dz}{\int \mu_p(z)\,dz} = c_p$$

If
$$\int \mu_p(z)\,dz = A_p$$

$$\Rightarrow \int z\, \mu_p(z)\,dz = A_p\, c_p$$



$$\int z\, \mu_p(z)\,dz = A_p\, c_p$$

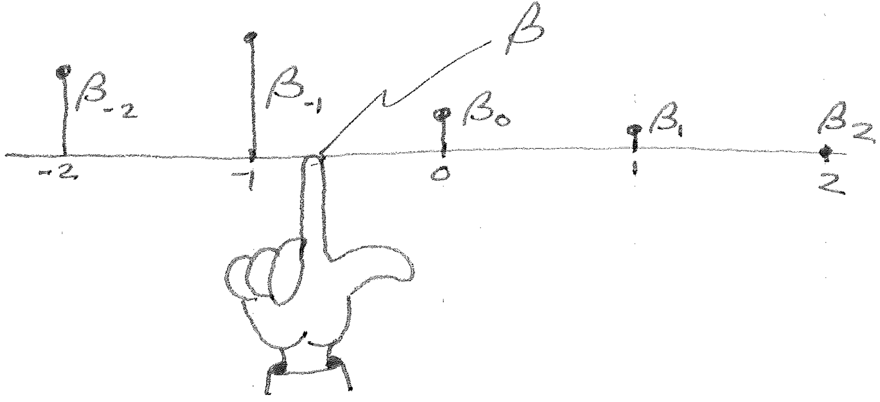
$$\Rightarrow \quad \bar{z} = \frac{\Sigma_p A_p\, c_p\, \beta_p}{\sum_p A_p \beta_p} \quad \longleftarrow \text{Always!}$$

Special case

$$A_p = A \quad \forall\, p \quad \ni \quad c_p = p\Delta$$

$$\bar{z} = \frac{\Delta \Sigma_p\, p\, \beta_p}{\sum_p \beta_p} = \Delta\, \bar{\beta}$$

where
$$\bar{\beta} = \frac{\Sigma_p\, p\, \beta_p}{\sum_p \beta_p}$$



For example on p.135

β

$$\bar{z} = \Delta \bar{\beta}$$

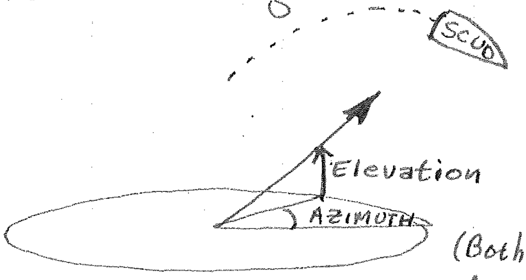
$$\bar{\beta} = \frac{\frac{1}{2} \cdot -1 + \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2}{\frac{1}{2} + 3/4 + \frac{1}{4}}$$

$$= 0 = \bar{z}$$

Recall, from cut, $\bar{z} = \frac{1}{17}$

Target Tracking

Problem: Adjust azimuth & elevation to point at target.



Control inputs:

e & $\dot{e}$

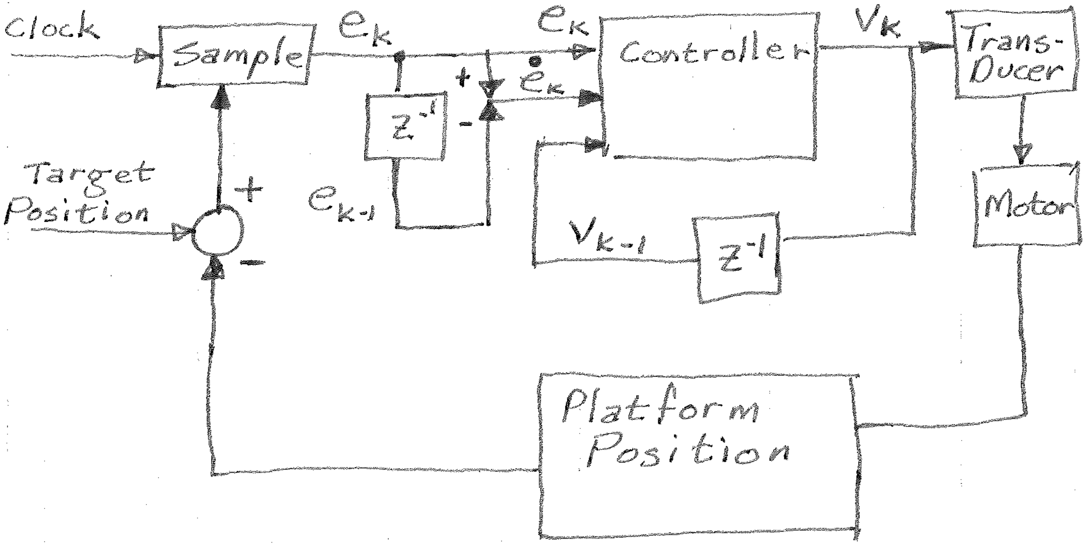
(Both Angles) Output

V_k = angular velocity

Note: Two separate controllers control azimuth & elevation. Their architecture will be identical.

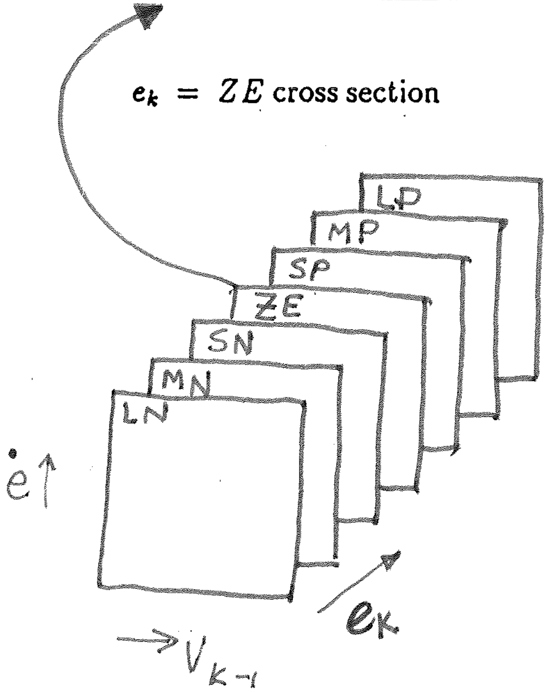
Target tracking system (z⁻¹ = unit delay)

($\dot{e}_k \approx e_k - e_{k-1}$



Pacini et al
'Adaptive Fuzzy System for Target Tracking'

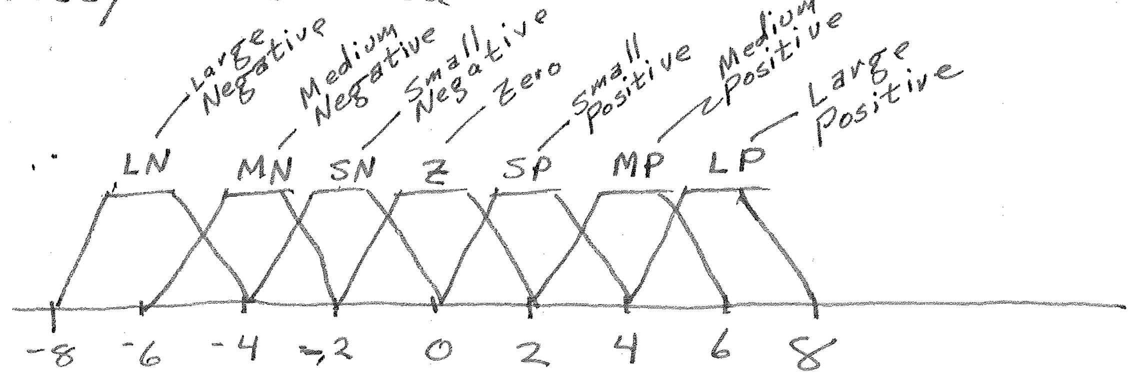
					V_{k-1}			
		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	MN	SN	ZE
	MN	LN	LN	LN	MN	SN	ZE	SP
	SN	LN	LN	MN	SN	ZE	SP	MP
$\dot{e}_k$	ZE	LN	MN	SN	ZE	SP	MP	LP
	SP	MN	SN	ZE	SP	MP	LP	LP
	MP	SN	ZE	SP	MP	LP	LP	LP
	LP	ZE	SP	MP	LP	LP	LP	LP



$e_k = ZE$ cross section

Controller will be fuzzy

Fuzzy Sets Used



In certain software & hardware, universe of discourse is fixed. ~~Va~~ Domain must be scaled to fit.

Fuzzification examples:

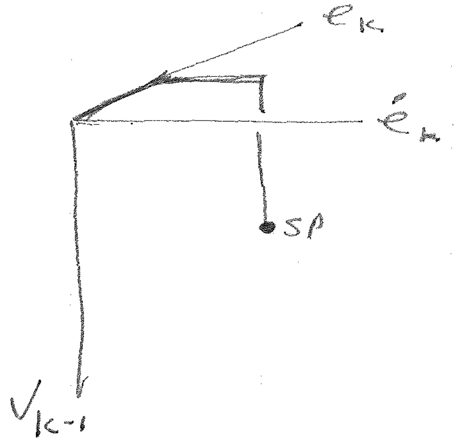
$1 \rightarrow (0\ 0\ 0\ .7\ .7\ 0\ 0)$
$-4 \rightarrow (0\ 1\ 0\ 0\ 0\ 0\ 0)$
$3.8 \rightarrow (0\ 0\ 0\ 0\ .1\ 1\ 0)$

Example Fuzzy Implication:

If $e_k = MP$ and $\dot{e}_k = SN$ and $V_{k-1} = ZE$

then $V_k = SP$

Note: Three dimensional table

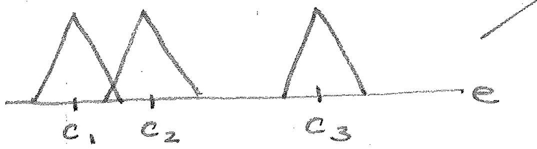


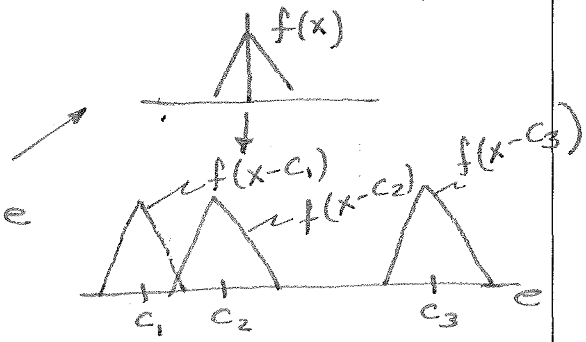
Sequential:

All membership functions have same shape, $f(x)$

∴ same area, $A_p = A$

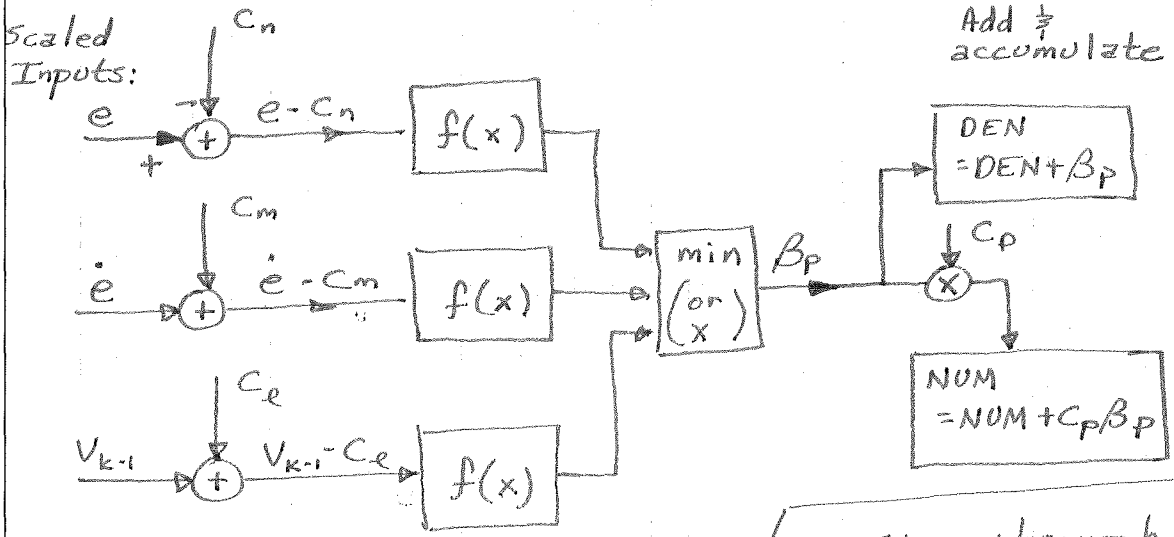
c_n = centroids of e

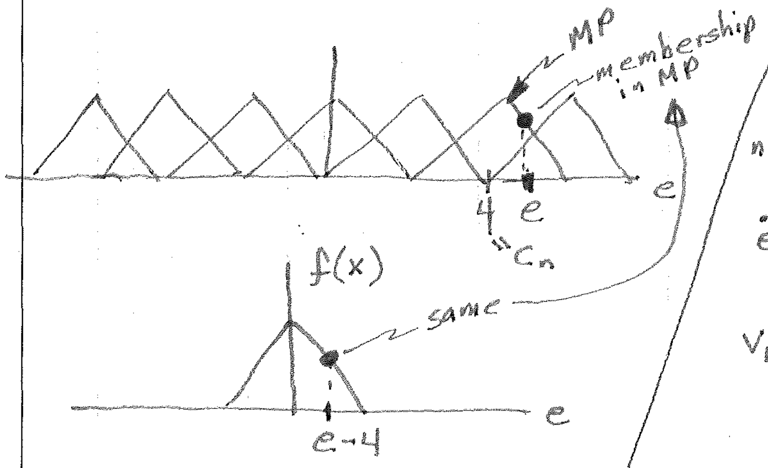




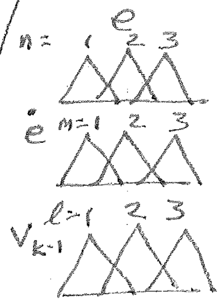
c_m = centroids of $\dot{e}$

c_ℓ = " " V_{k-1}





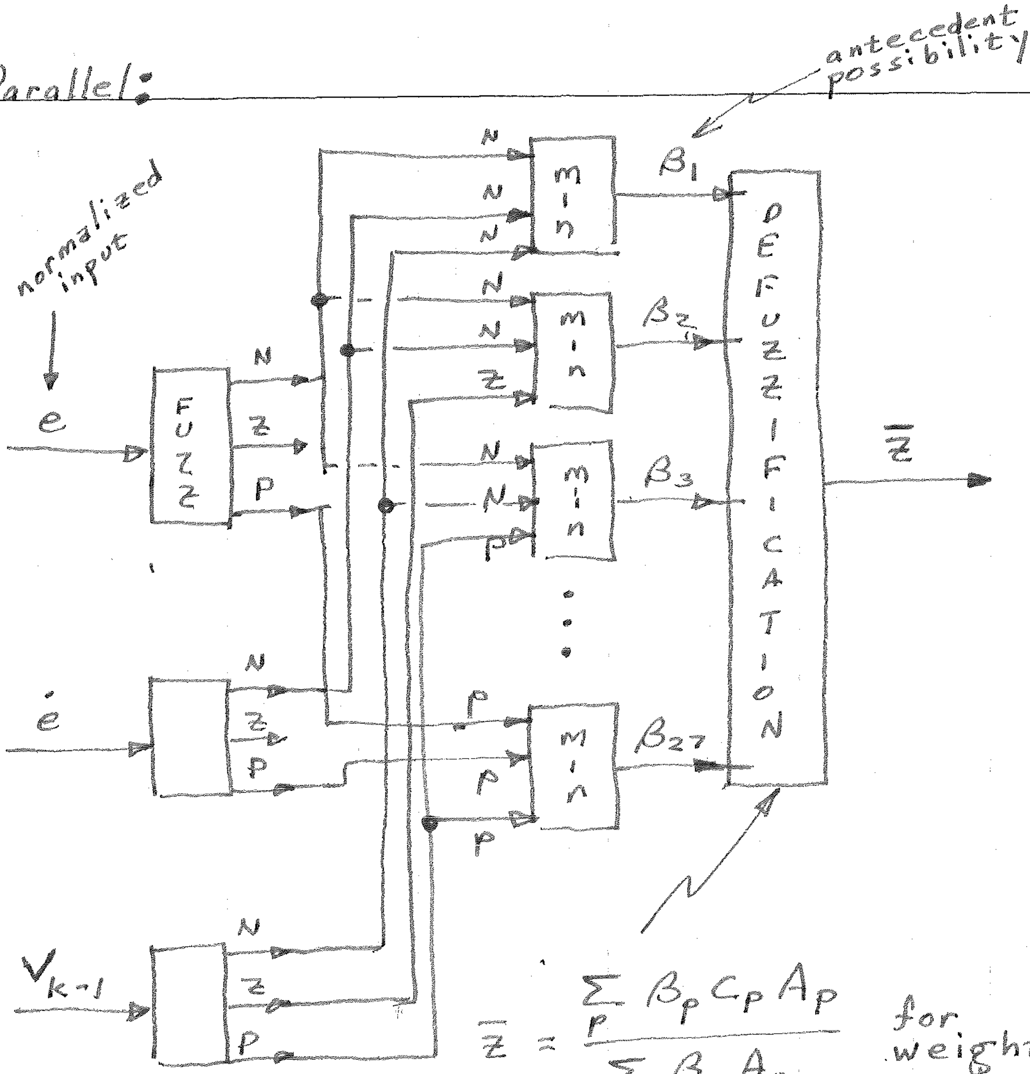
step through inputs in sawtooth:



~~ℓ_n~~

n m ℓ	p
1 1 1	1
1 1 2	2
1 1 3	3
1 2 1	4
1 2 2	5
1 2 3	6
1 3 1	7
⋮	
3 3 3	27

Parallel:

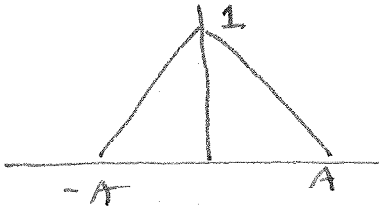


$$\bar{z} = \frac{\sum_p \beta_p C_p A_p}{\sum_p \beta_p A_p}$$ for weighted membership

Note: This is alternate type of defuzzification. There is no OR combination of anticedents for a a common consequent.

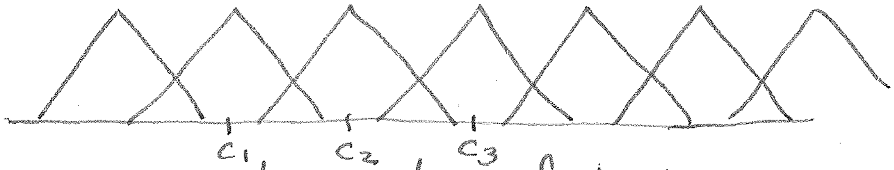
Obvious application to

- cut membership functions
- $f(x) = \left(1 - \frac{|x|}{A}\right)\Pi\left(\frac{x}{2A}\right)$



- Using single f(x)
  (trade less space for more time)

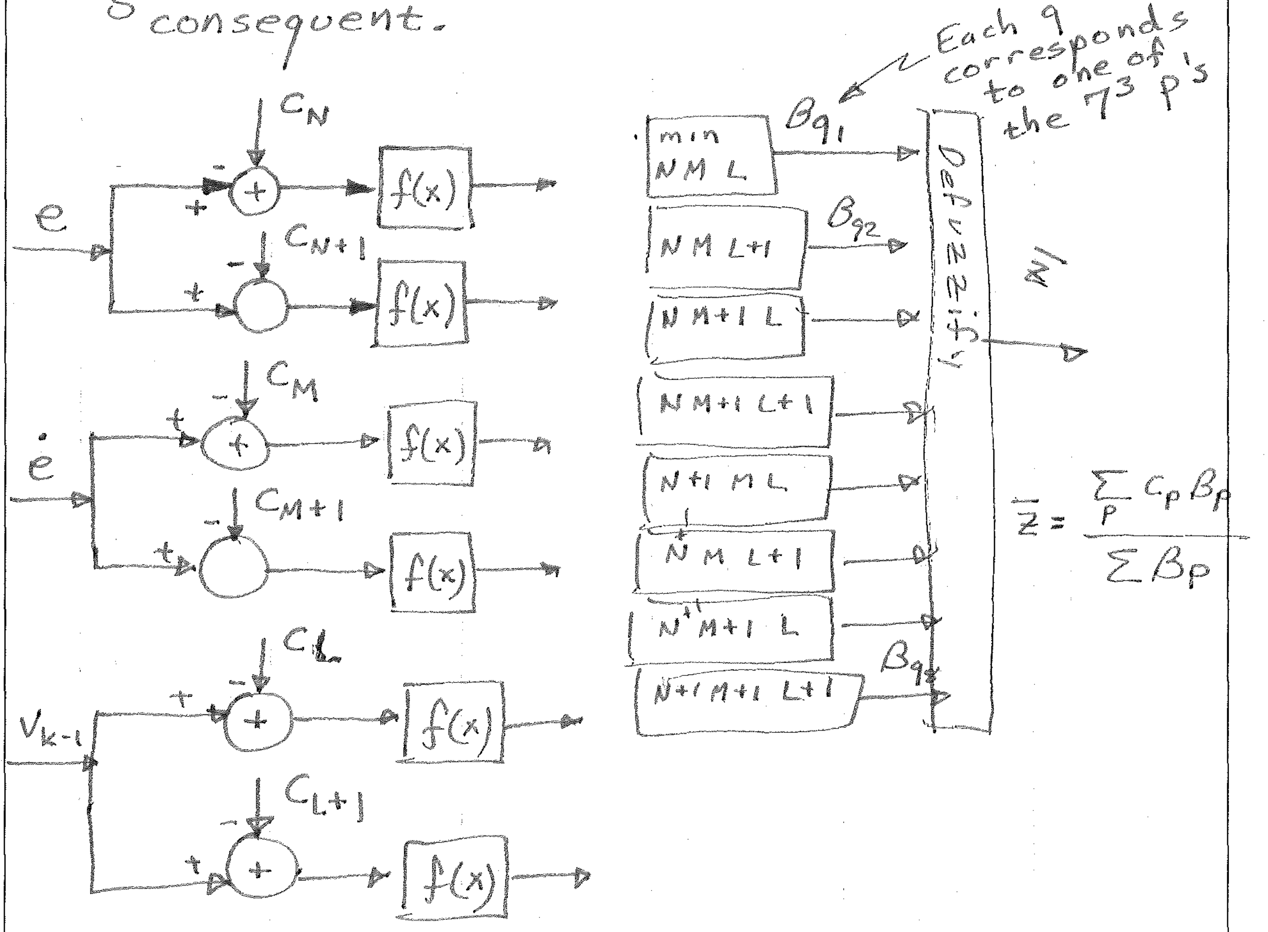
Note: Most β_p's are zero's if



Assume maximal overlap of two membership functions

Given e, find unique $n = N \ni c_N < \epsilon < c_{N+1}$
" $\dot{e}$, " " $m = M \ni c_M < \epsilon < c_{M+1}$
" V_{k-1}, " " $\ell = L \ni c_L < V_{k-1} < c_{L+1}$

Only eight $(= 2^3)$ antecedents will give a nonzero contribution to consequent.



$$\bar{z} = \frac{\sum_p c_p \beta_p}{\sum \beta_p}$$

Note:
Front end: Find N, M, L & corresponding C's
Back end: match q's to p's.

In general, for V variables in antecedant, 2^V mins are required.

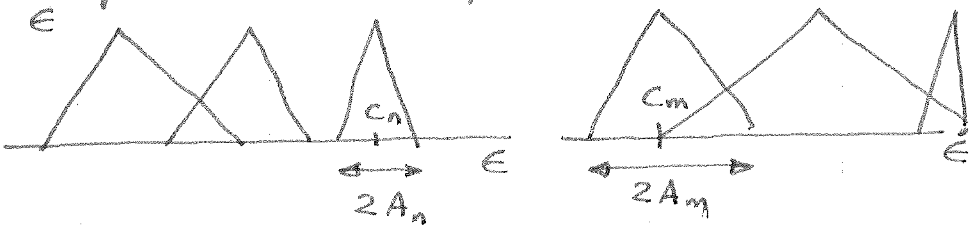
★ Adapting membership functions & fuzzy controllers

PROBLEM: A fuzzy control system is designed. In the field, it works 'well'. Can we fine tune the system to make it work better given it's performance on real data? ie, can we make it adaptive?

We will illustrate with adaption of the input fuzzy membership functions.

Rules: If $\overset{\epsilon}{=} \mathbb{A}_n$ and $\overset{\dot{\epsilon}}{=} \mathbb{B}_m$, then $\mathbb{C}_p$

Assume input membership functions:



Based on 'performance', we will adjust c_n, A_n, c_m & A_m's.

Performance:

For crisp input $(e, \dot{e})$, let output of fuzzy controller be

$$\overline{z}(e, \dot{e})$$

Desired (target) output is

$$t(e, \dot{e})$$

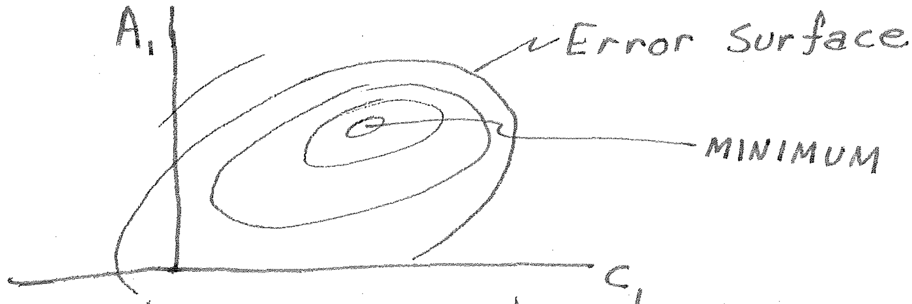
Error:

$$E(e, \dot{e}) = \frac{1}{2}\left(\overline{z}(e, \dot{e}) - t(e, \dot{e})\right)^2$$

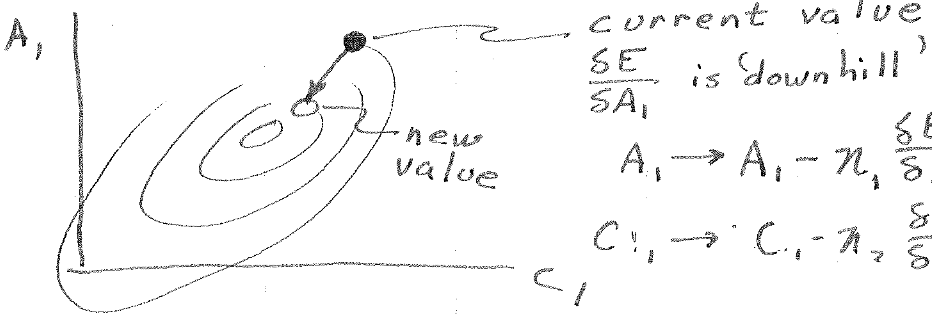
Note that $\overline{z}$ (and thus E) is a function of the c's & A's.

Problem: Find c's & A's such that E is minimized.

A 2-D illustration



∃ a number of search techniques to find a minimum. We choose 'steepest descent'



$\frac{\delta E}{\delta A_1}$ is 'downhill'

$$A_1 \rightarrow A_1 - \eta_1 \frac{\delta E}{\delta A_1}$$

$$C_1 \rightarrow C_1 - \eta_2 \frac{\delta E}{\delta a_1}$$

η = step size

Steepest descent used in modems, adaptive filters, adaptive optics, bp training of nn's.

Q: How do we find $\frac{\delta E}{\delta A}$'s and $\frac{\delta E}{\delta C}$'s?

A: Error backpropagation.

$$a_n \longrightarrow a_n - \eta \frac{\delta E}{\delta \;{}_{n}}$$

Similar update can be given for A_n's

${}_m$'s

A_m's

Also!

We can update

- output membership functions
- if-then rules

Let

$$E = \frac{1}{2}\left(\bar{Z} - t\right)^2$$

$$\frac{\delta E}{\delta C_n} = \left(\bar{Z} - t\right)\frac{\delta \bar{Z}}{\delta \epsilon_n}$$

$$\bar{Z} = \frac{\Sigma_{\hat{p}}\, \beta_{\hat{p}}\, c_{\hat{p}}\, A_{\hat{p}}}{\Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}}$$

ONLY β_p is a function of a_n

$$\frac{\delta \bar{Z}}{\delta C_n} = \sum_p \frac{\delta \bar{Z}}{\delta \beta_p}\; \frac{\delta \beta_p}{\delta C_n}$$

$$\frac{\delta \bar{Z}}{\delta \beta_p} = \frac{\Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}\, c_p A_p - A_p\, \Sigma_{\hat{p}}\, \beta_{\hat{p}} c_{\hat{p}} A_{\hat{p}}}{\left(\Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}\right)^2}$$

$$= \frac{A_p\, \Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}\left(c_p - c_{\hat{p}}\right)}{\left(\Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}\right)^2}$$

Using product:

$$\beta_p = \mu_{\hat{n}}(\epsilon)\, \mu_m^{(p)}(\dot{\epsilon})$$

Assume $p^{\underline{th}}$ consequent is a result only of $(n,m)^{\underline{th}}$ Antecetant

$$\frac{\delta \beta_p}{\delta C_n} = \mu_m(\dot{\epsilon})\; \frac{\delta \mu_n(\epsilon)}{\delta C_n}$$

Let $\Lambda(x) = \left(1 - |x|\right)\Pi\left(\frac{x}{2}\right)$ (triangle)

and

$$\mu_n(\epsilon) = \Lambda\left(\frac{\epsilon - C_n}{2 A_n}\right)$$

$$\frac{\delta \mu_n(\epsilon)}{\delta C_n} = \frac{-1}{2 A_n}\, \Lambda'\left(\frac{\epsilon - C_n}{2 A_n}\right)$$

$$\Lambda'(x) = -\, \mathrm{sgn}(x)\, \Pi\left(\frac{x}{2}\right)$$

Thus

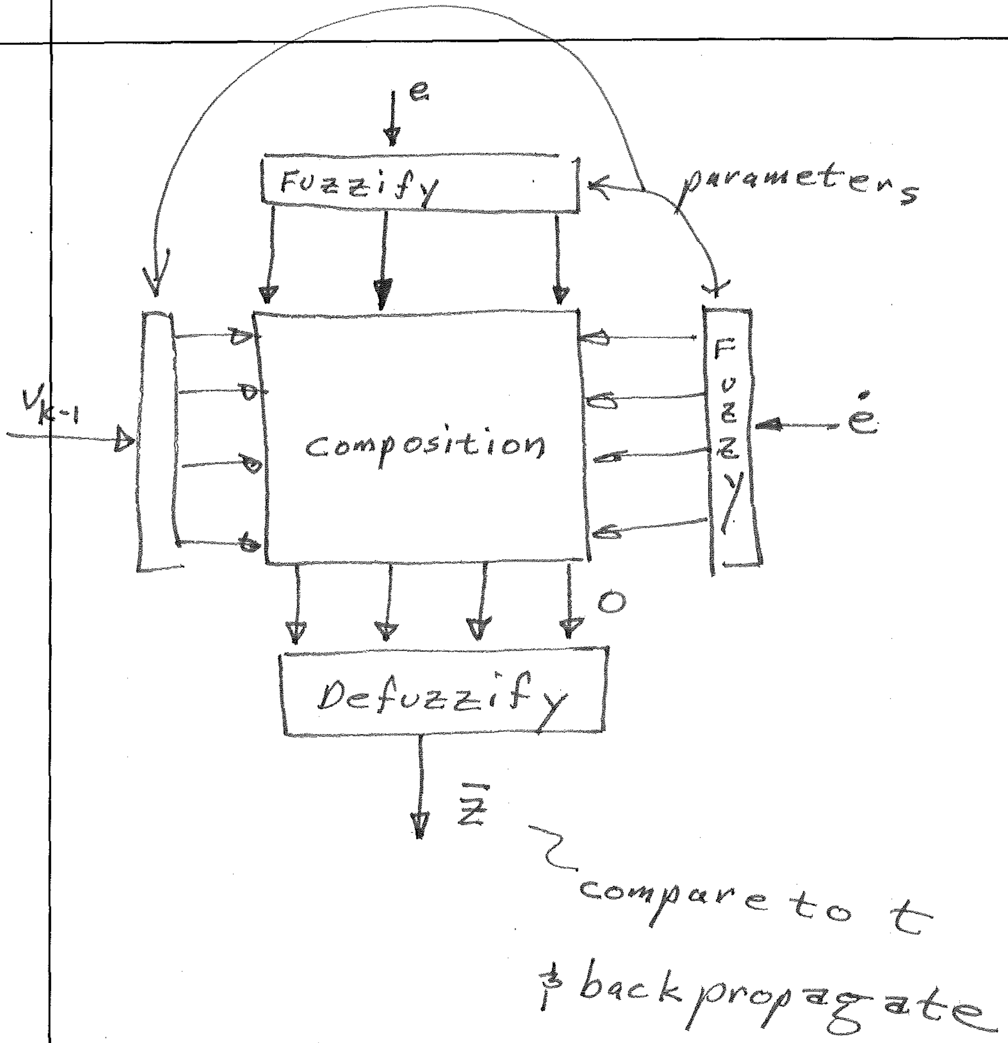
$$\frac{\delta E}{\delta a_n} = \left(\bar{Z} - t\right)\frac{\delta \bar{Z}}{\delta C_n} = \left(\bar{Z} - t\right)\sum_p \frac{\delta \bar{Z}}{\delta \beta_p}\, \frac{\delta \beta_p}{\delta C_n}$$

$$= \left(\bar{Z} - t\right)\sum_p \frac{\delta \bar{Z}}{\delta \beta_p}\, \mu_m(\dot{\epsilon})\, \frac{\delta \mu_n(\epsilon)}{\delta C_n}$$

$$= \left(\bar{Z} - t\right)\sum_p \left[\frac{A_p\, \Sigma_{\hat{p}}\, \beta_p A_{\hat{p}}\left(c_p - c_{\hat{p}}\right)}{\left(\Sigma_{\hat{p}}\, \beta_{\hat{p}} A_{\hat{p}}\right)^2}\right]\mu_m(\dot{\epsilon})$$

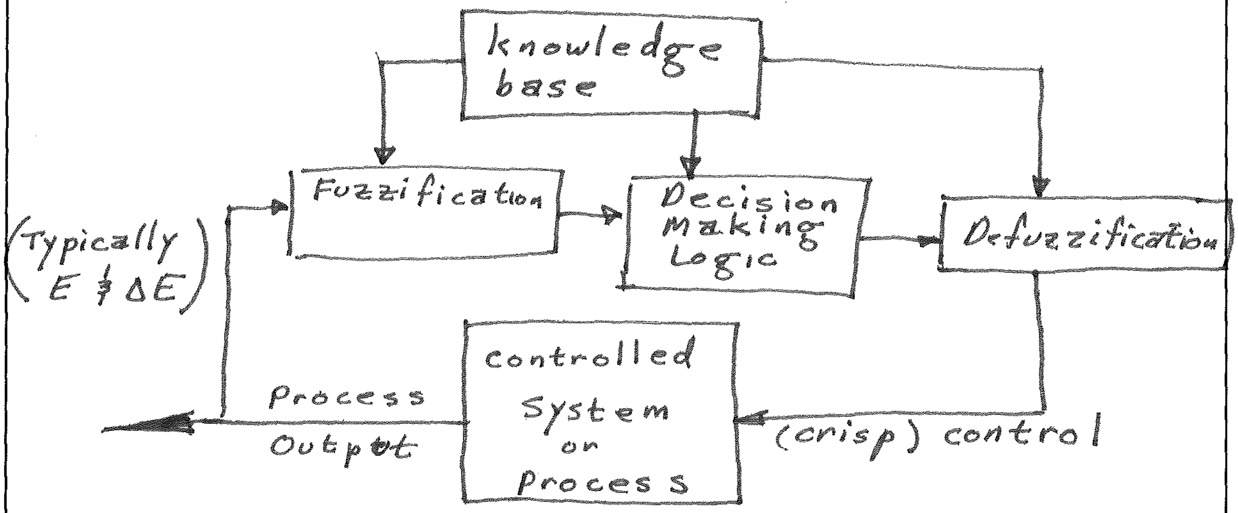
$$\times\; \frac{+1}{2 A_n}\, \mathrm{sgn}\left(\frac{\epsilon - C_n}{2 A_n}\right)\Pi\left(\frac{\epsilon - C_n}{4 A_n}\right)$$

$= \pm 1$ or zero


e
Fuzzify
parameters
v_{k-1}
Composition
Fuzzy
$\dot{e}$
0
Defuzzify
$\bar{z}$
compare to t
& backpropagate

Lee's Tutorial

Basic Configuration of fuzzy logic controller



knowledge base:

- define rules,
  - membership functions (fuzzy partition)
  - control policy

  } Experts

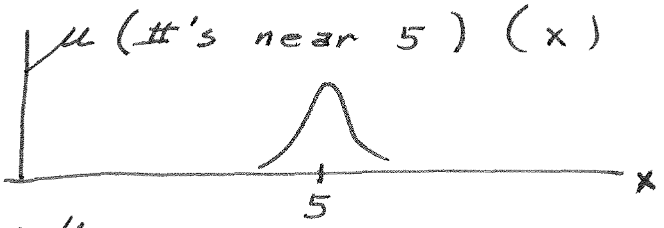
  - universes of discourses

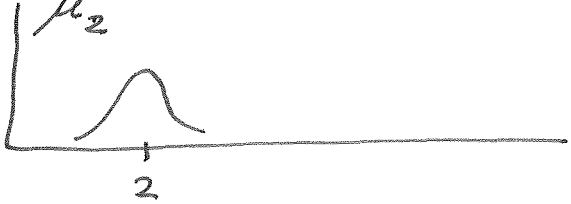
Where do we get our rules?

1. Experts Experience & Control Eng. Knowledge
2. Operator Control Actions
3. Fuzzy Model of A Process
4. Learned
   'Sugeno's fuzzy car can be trained to park by itself'

Fuzzy Arithmitic

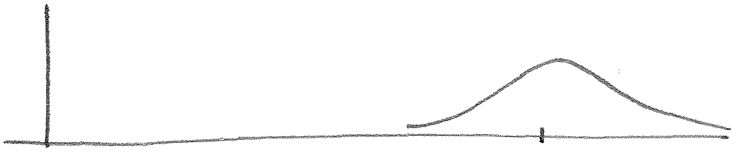
Fuzzy Number





Recall Convolution

$$\mu_5 * \mu_2 = \mu_7$$

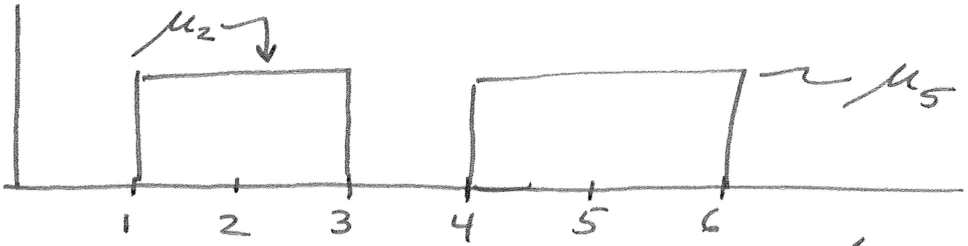


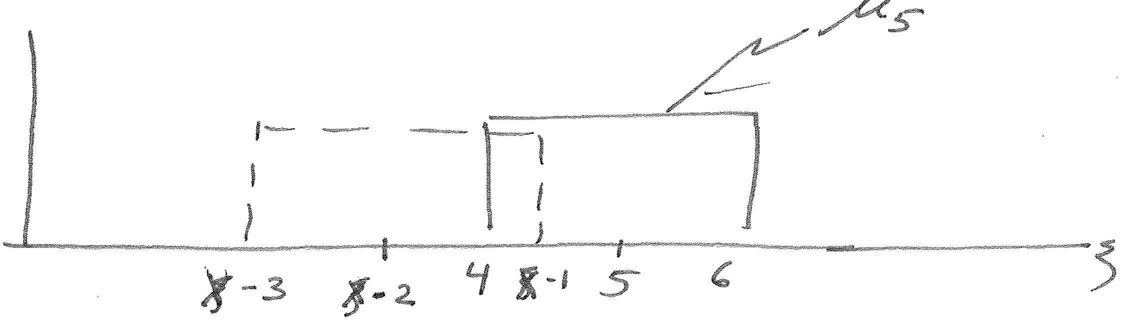
$$\mu_7(x) = \int_{-\infty}^{\infty} \mu_5(\xi)\mu_2(x-\xi)\,d\xi$$

FUZZY CONVOLUTION

$$\mu_7(x) = \max \min_{\xi} \mu_5(\xi)\mu_2(x-\xi)$$

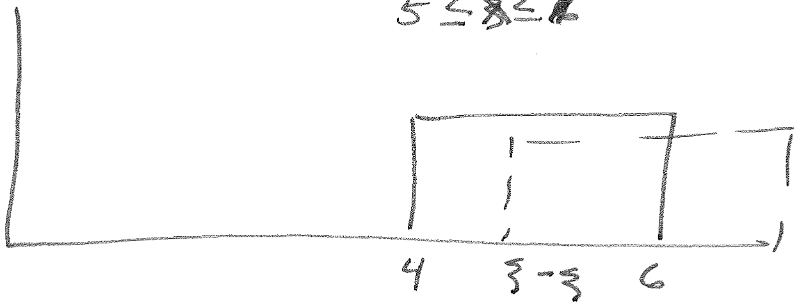
~~Try crisp:~~





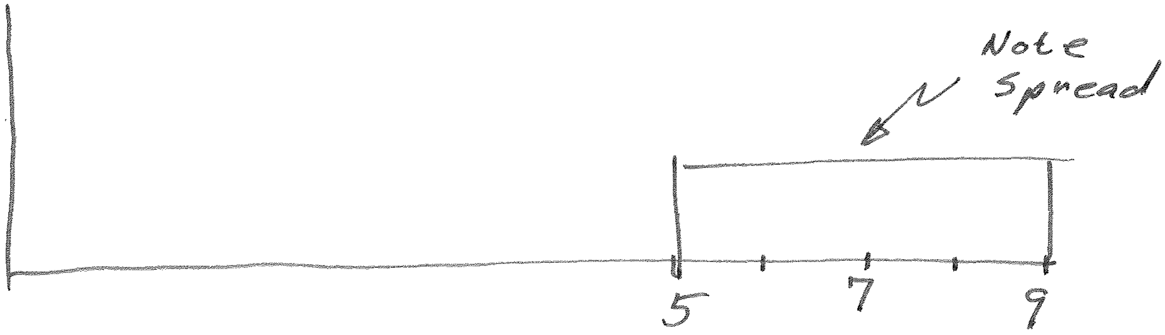
For $4 \leq \xi - 1 \leq 6 \Rightarrow 1$

$5 \leq \xi \leq 7$



$4 \leq \xi - 3 \leq 6$

$7 \leq \xi \leq 9 \Rightarrow 1$



Note: Adding two triangular functions results in a triangular function.

more general

$$\mu_{A+B}(z) = \bigvee_{z=x+y} \left[\mu_A(x) \wedge \mu_B(y) \right]$$

$$= \bigvee \left[\mu_A(x) \wedge \mu_B(z-x) \right]$$

Multipication:

$$\mu_{A\cdot B}(z) = \bigvee_{z=x\cdot y} \left[\mu_A(x) \wedge \mu_B(y) \right]$$

$$= \bigvee \quad \mu_A(x) \wedge \mu_B(z/x)$$

For min/max

$$\mu_{A\cdot B}(z) = \max \quad \min \mu_A(x) \, \mu_B(z/x)$$

Note: Can show

$$\underbrace{(A+B)\cdot C = (A\cdot C) + (B\cdot C)}_{\text{Fuzzy operations}}$$

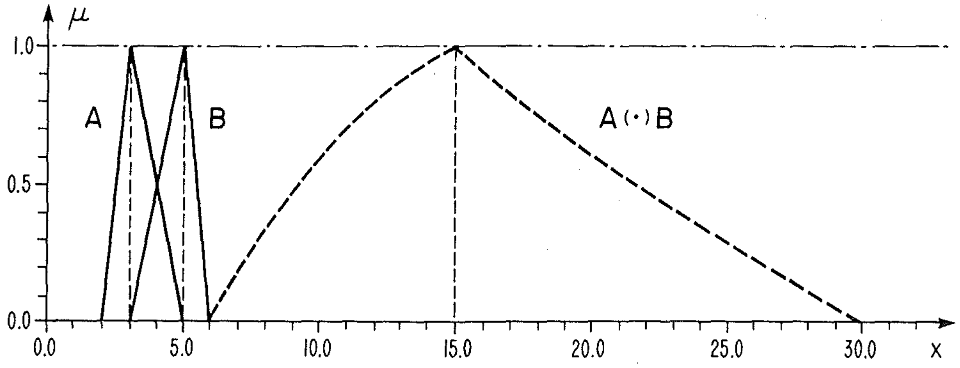


Figure 1.12 Multiplication of two fuzzy numbers (Example 1.9).

and

$$\alpha = -a_2^{(\alpha)}/2 + 5/2.$$

Hence

$$A_\alpha = [\alpha + 2, -2\alpha + 5].$$

Using (1.44) we also have

$$\alpha = b_1^{(\alpha)}/2 - 3/2 \qquad \text{and} \qquad \alpha = -b_2^{(\alpha)} + 6. \text{ Hence}$$

$$B_\alpha = [2\alpha + 3, -\alpha + 6].$$

Thus we obtain the multiplication

$$\begin{aligned} A_\alpha (\cdot)\, B_\alpha &= [\alpha + 2)\,(2\alpha + 3), (-2\alpha + 5)\,(-\alpha + 6)] \\ &= [2\alpha^2 + 7\alpha + 6,\ 2\alpha^2 - 17\alpha + 30]. \end{aligned}$$

We now have two equations to solve, namely,

$$2\alpha^2 + 7\alpha + 6 - x = 0 \tag{1.46}$$

and

$$2\alpha^2 - 17\alpha + 30 - x = 0. \tag{1.47}$$

We will retain only two roots in [0, 1]. For (1.46)

$$\alpha = (-7 + \sqrt{1 + 8x})/4,$$

and for (1.47)

$$\alpha = (17 - \sqrt{49 + 8x})/4.$$

Finally,

$\forall\, x \in R^+$:

The resulti

A (·) **B** doe

EXAMPLE

In N let us

Using (1.41)
thus find th

A (·) **B** =

8	9	10
0	0.1	0.1

20	21
1	0.7

We now exan
we showed h
number in R
of the norma
of the norma

★ 1. At the left
in (1.42),
★ 2. At the rig
To simplif

Division:

$$\mu_{A \div B}(z) = \bigvee_{z = \frac{x}{y}} \left[\mu_A(x) \wedge \mu_B(y) \right]$$

$$= \bigvee \quad \mu_A(x) \wedge \mu_B(x/z)$$

$$= \bigvee_{y} \quad \mu_A(yz) \wedge \mu_B(y)$$

$$= \max \overset{\min}{{}^{\vee}} \mu_A(x) \; \mu_B(x/z)$$

$$\mu_{\mathbf{A}(-)\mathbf{B}}(1) = (0.3 \wedge 0) \vee (0.7 \wedge 0.1) \vee (0.9 \wedge 0.6) \vee (1 \wedge 1) \vee (0.5 \wedge 0.8) = 1,$$

$$\mu_{\mathbf{A}(-)\mathbf{B}}(2) = (0.7 \wedge 0) \vee (0.9 \wedge 0.1) \vee (1 \wedge 0.6) \vee (0.5 \wedge 0.8) = 0.6,$$

$$\mu_{\mathbf{A}(-)\mathbf{B}}(3) = (0.9 \wedge 0) \vee (1 \wedge 0.1) \vee (0.5 \wedge 0.6) = 0.5,$$

$$\mu_{\mathbf{A}(-)\mathbf{B}}(4) = (1 \wedge 0) \vee (0.5 \wedge 0.1) = 0.1,$$

$$\mu_{\mathbf{A}(-)\mathbf{B}}(5) = (0.5 \wedge 0) = 0.$$

1.5 MULTIPLICATION OF FUZZY NUMBERS

At this point we consider multiplication[5] in R^+ and N. Let us consider two fuzzy numbers **A** and **B** in R^+. From the level α of presumption, we can write

$$\begin{aligned} A_\alpha (\cdot) B_\alpha &= [a_1^{(\alpha)}, a_2^{(\alpha)}] (\cdot) [b_1^{(\alpha)}, b_2^{(\alpha)}] \\ &= [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}]. \end{aligned} \qquad \textbf{(1.41)}$$

Multiplication can also be given by

$\forall\, x, y, z \in R^+$:

$$\mu_{\mathbf{A}(\cdot)\mathbf{B}}(z) = \bigvee_{z = x \cdot y} (\mu_{\mathbf{A}}(x) \wedge \mu_{\mathbf{B}}(y)). \qquad \textbf{(1.42)}$$

Equations (1.41) and (1.42) are equivalent, and this may be proven in the same way as was given for addition in the previous section.

EXAMPLE 1.9

For this example we again use triangular fuzzy numbers because they are so easy to work with.

$\forall\, x \in R^+$:

$$\begin{aligned} \mu_{\mathbf{A}}(x) &= 0, & x &\le 2, \\ &= x - 2, & 2 \le x &\le 3, \\ &= -x/2 + 5/2, & 3 \le x &\le 5, \\ &= 0, & x &\ge 5. \end{aligned} \qquad \textbf{(1.43)}$$

$$\begin{aligned} \mu_{\mathbf{B}}(x) &= 0, & x &\le 3, \\ &= x/2 - 3/2, & 3 \le x &\le 5, \\ &= -x + 6, & 5 \le x &\le 6, \\ &= 0, & x &\ge 6. \end{aligned} \qquad \textbf{(1.44)}$$

For the level α in Figure 1.12 and using (1.43) we have

$$\alpha = a_1^{(\alpha)} - 2, \qquad \textbf{(1.45)}$$

$$B_\alpha^{-1} = [1/b_2^{(\alpha)},\ 1/b_1^{(\alpha)}], \qquad b_2^{(\alpha)} > 0, \qquad \forall\ \alpha \in [0, 1]. \quad \textbf{(1.50)}$$

Division is not, however, associative or commutative. In connection with this let us study an example in R^+.

EXAMPLE 1.11

We now consider a numerical example. Let us use the triangular shape shown in Figure 1.13, and let

$\forall\ x \in R^+$:

$$\begin{aligned}
\mu_{\mathbf{A}}(x) &= 0, & x &\le 18,\\
&= x/4 - 18/4, & 18 \le x &\le 22,\\
&= -x/11 + 3, & 22 \le x &\le 33,\\
&= 0, & x &\ge 33. & \textbf{(1.51)}\\
\mu_{\mathbf{B}}(x) &= 0, & x &\le 5,\\
&= x - 5, & 5 \le x &\le 6,\\
&= -x/2 + 4, & 6 \le x &\le 8,\\
&= 0, & x &\ge 8. & \textbf{(1.52)}
\end{aligned}$$

In (1.51), let $\alpha = a_1^{(\alpha)}/4 - 18/4$ and $\alpha = -a_2^{(\alpha)}/11 + 3$, from which

$$A_\alpha = [4\alpha + 18,\ -11\alpha + 33].$$

In (1.52), let $\alpha = b_1^{(\alpha)} - 5$ and $\alpha = -b_2^{(\alpha)}/2 + 4$, from which

$$B_\alpha = [\alpha + 5,\ -2\alpha + 8].$$

Thus

$$\begin{aligned}
A_\alpha\ (:)\ B_\alpha &= [4\alpha + 18,\ -11\alpha + 33]\ (:)\ [\alpha + 5,\ -2\alpha + 8]\\
&= \left(\frac{4\alpha + 18}{-2\alpha + 8},\ \frac{-11\alpha + 33}{\alpha + 5}\right)
\end{aligned}$$

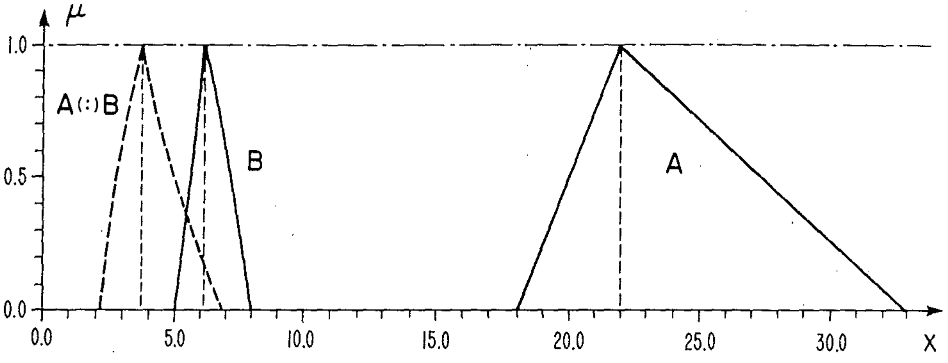


Figure 1.13 Division of two fuzzy numbers (Example 1.11).

Notes:

- Operations 'twixt Fuzzy numbers & Crisp numbers yield what you think they do
- Operations can be performed on α cuts.
- Applications
  - Finance
  - Forecasting
  - etc.

Fuzzy Logic in Control Systems: Fuzzy Logic Controller—Part I

CHUEN CHIEN LEE, STUDENT MEMBER, IEEE

Abstract —**During the past several years, fuzzy control has emerged as one of the most active and fruitful areas for research in the applications of fuzzy set theory, especially in the realm of industrial processes, which do not lend themselves to control by conventional methods because of a lack of quantitative data regarding the input-output relations. Fuzzy control is based on fuzzy logic—a logical system which is much closer in spirit to human thinking and natural language than traditional logical systems. The fuzzy logic controller (FLC) based on fuzzy logic provides a means of converting a linguistic control strategy based on expert knowledge into an automatic control strategy. A survey of the FLC is presented; a general methodology for constructing an FLC and assessing its performance is described; and problems that need further research are pointed out. In particular, the exposition includes a discussion of fuzzification and defuzzification strategies, the derivation of the database and fuzzy control rules, the definition of fuzzy implication, and an analysis of fuzzy reasoning mechanisms.**

I. INTRODUCTION

DURING the past several years, fuzzy control has emerged as one of the most active and fruitful areas for research in the application of fuzzy set theory [141]. The pioneering research of Mamdani and his colleagues on fuzzy control [63]–[66], [50] was motivated by Zadeh's seminal papers on the linguistic approach and system analysis based on the theory of fuzzy sets [142], [143], [145], [146]. Recent applications of fuzzy control in water quality control [127], [35], automatic train operation systems [135], [136], [139], automatic container crane operation systems [137]–[139], elevator control [23], nuclear reactor control [4], [51], automobile transmission control [40], fuzzy logic controller hardware systems [130], [131], fuzzy memory devices [107], [108], [120], [128], [129], [133], and fuzzy computers [132] have pointed a way for an effective utilization of fuzzy control in the context of complex ill-defined processes that can be controlled by a skilled human operator without the knowledge of their underlying dynamics.

The literature in fuzzy control has been growing rapidly in recent years, making it difficult to present a comprehensive survey of the wide variety of applications that have been made. Historically, the important milestones in the development of fuzzy control may be summarized as shown in table I. It should be stressed, however, the choice of the milestones is a subjective matter.

Fuzzy logic, which is the logic on which fuzzy control is based, is much closer in spirit to human thinking and natural language than the traditional logical systems. Basically, it provides an effective means of capturing the approximate, inexact nature of the real world. Viewed in this perspective, the essential part of the fuzzy logic controller (FLC) is a set of linguistic control rules related by the dual concepts of fuzzy implication and the compositional rule of inference. In essence, then, the FLC provides an algorithm which can convert the linguistic control strategy based on expert knowledge into an automatic control strategy. Experience shows that the FLC yields results superior to those obtained by conventional control algorithms. In particular, the methodology of the FLC appears very useful when the processes are too complex for analysis by conventional quantitative techniques or when the available sources of information are interpreted qualitatively, inexactly, or uncertainly. Thus fuzzy logic control may be viewed as a step toward a rapprochement between conventional precise mathematical control and human-like decision making, as indicated by Gupta [30].

However, at present there is no systematic procedure for the design of an FLC. In this paper we present a survey of the FLC methodology and point to the problems which need further research. Our investigation includes fuzzification and defuzzification strategies, the derivation of the database and fuzzy control rules, the definition of a fuzzy implication, and an analysis of fuzzy reasoning mechanisms.

This paper is divided into two parts. The analysis of structural parameters of the FLC is addressed in Part I. In addition, Part I contains five more sections. A brief summary of some of the relevant concepts in fuzzy set theory and fuzzy logic is presented in Section II. The main idea of the FLC is described in Section III, while Section IV describes the fuzzification strategies. In Section V, we discuss the construction of the data base of an FLC. The rule base in Section VI explains the derivation of fuzzy control rules and rule-modification techniques.

Part II consists of four sections. Section I is devoted to the basic aspects of the FLC decision-making logic. Several issues including the definitions of a fuzzy implication,

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TABLE I

1972	Zadeh	A rationale for fuzzy control [145]
1973	Zadeh	Linguistic approach [146]
1974	Mamdani & Assilian	Steam engine control [64]
1976	Rutherford *et al.*	Analysis of control algorithms [5], [7]
1977	Ostergaard	Heat exchanger and cement kiln control [80]
1977	Willaeys *et al.*	Optimal fuzzy control [121]
1979	Komolov *et al.*	Finite automaton [57]
1980	Tong *et al.*	Wastewater treatment process [113]
1980	Fukami, Mizumoto and Tanaka	Fuzzy conditional inference [24]
1983	Hirota and Pedrycz	Probabilistic fuzzy sets (control) [33]
1983	Takagi and Sugeno	Derivation of fuzzy control rules [103]
1983	Yasunobu, Miyamoto *et al.*	Predictive fuzzy control [135]
1984	Sugeno and Murakami	Parking control of a model car [97]
1985	Kiszka, Gupta *et al.*	Fuzzy system stability [55]
1985	Togai and Watanabe	Fuzzy chip [107]
1986	Yamakawa	Fuzzy controller hardware system [130]
1988	Dubois and Prade	Approximate reasoning [21]

compositional operators, the interpretations of sentence connectives "and" and "also," and fuzzy inference mechanisms, are investigated. Section II discusses the defuzzification strategies. Some of the representative applications of the FLC, from laboratory level to industrial process control, are briefly reported in Section III. Finally, we describe some unsolved problems and discuss further challenges in this field.

II. FUZZY SETS AND FUZZY LOGIC

For the convenience of the reader, we shall briefly summarize some of the basic concepts of fuzzy set theory and fuzzy logic which will be needed in this paper. A more detailed discussion may be found in [141], [41], [42], [148], [149] and [21].

A. Fuzzy Sets and Terminology

Let U be a collection of objects denoted generically by $\{u\}$, which could be discrete or continuous. U is called the universe of discourse and u represents the generic element of U.

Definition 1: Fuzzy Set: A fuzzy set F in a universe of discourse U is characterized by a membership function μ_F which takes values in the interval $[0,1]$ namely, μ_F: $U \rightarrow [0,1]$. A fuzzy set may be viewed as a generalization of the concept of an ordinary set whose membership function only takes two values $\{0,1\}$. Thus a fuzzy set F in U may be represented as a set of ordered pairs of a generic element u and its grade of membership function: $F = \{(u, \mu_F(u)) | u \in U\}$. When U is continuous, a fuzzy set F can be written concisely as $F = \int_U \mu_F(u)/u$. When U is discrete, a fuzzy set F is represented as

$$F = \sum_{i=1}^{n} \mu_F(u_i)/u_i.$$

Definition 2: Support, Crossover Point, and Fuzzy Singleton: The support of a fuzzy set F is the crisp set of all points u in U such that $\mu_F(u) > 0$. In particular, the element u in U at which $\mu_F = 0.5$, is called the crossover point and a fuzzy set whose support is a single point in U with $\mu_F = 1.0$ is referred to as fuzzy singleton.

B. Set Theoretic Operations

Let A and B be two fuzzy sets in U with membership functions μ_A and μ_B, respectively. The set theoretic operations of union, intersection and complement for fuzzy sets are defined via their membership functions. More specifically, see the following.

Definition 3: Union: The membership function $\mu_{A \cup B}$ of the union $A \cup B$ is pointwise defined for all $u \in U$ by

$$\mu_{A \cup B}(u) = \max\{\mu_A(u), \mu_B(u)\}.$$

Definition 4: Intersection: The membership function $\mu_{A \cap B}$ of the intersection $A \cap B$ is pointwise defined for all $u \in U$ by

$$\mu_{A \cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}.$$

Definition 5: Complement: The membership function $\mu_{\bar{A}}$ of the complement of a fuzzy set A is pointwise defined for all $u \in U$ by

$$\mu_{\bar{A}}(u) = 1 - \mu_A(u).$$

Definition 6: Cartesian Product: If $A_1, \cdots, A_n$ are fuzzy sets in $U_1, \cdots, U_n$, respectively, the Cartesian product of $A_1, \cdots, A_n$ is a fuzzy set in the product space $U_1 \times \cdots \times U_n$ with the membership function

$$\mu_{A_1 \times \cdots \times A_n}(u_1, u_2, \cdots u_n) = \min\{\mu_{A_1}(u_1), \cdots, \mu_{A_n}(u_n)\}$$

or

$$\mu_{A_1 \times \cdots \times A_n}(u_1, u_2, \cdots, u_n) = \mu_{A_1}(u_1) \cdot \mu_{A_2}(u_2) \cdots \mu_{A_n}(u_n).$$

Definition 7: Fuzzy Relation: An n-ary fuzzy relation is a fuzzy set in $U_1 \times \cdots \times U_n$ and is expressed as

$$R_{U_1 \times \cdots \times U_n} = \{((u_1, \cdots, u_n), \mu_R(u_1, \cdots, u_n)) | (u_1, \cdots, u_n) \in U_1 \times \cdots \times U_n\}.$$

Definition 8: Sup-Star Composition: If R and S are fuzzy relations in $U \times V$ and $V \times W$, respectively, the composition of R and S is a fuzzy relation denoted by

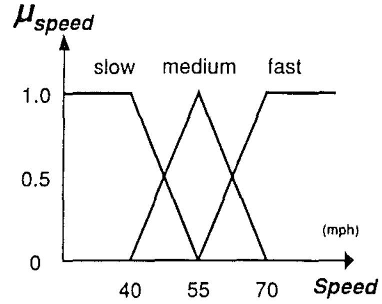


Fig. 1. Diagrammatic representation of fuzzy speeds. "Speed" is linguistic variable with three terms: "slow," "medium," and "high."

$R \circ S$ and is defined by

$$R \circ S = \left\{ \left[(u,w), \sup_{v} (\mu_R(u,v) * \mu_s(v,w)) \right], \quad u \in U, v \in V, w \in W \right\}$$

where $*$ could be any operator in the class of triangular norms, namely, minimum, algebraic product, bounded product, or drastic product (also see Part II [150]).

C. Linguistic Variables and Fuzzy Sets

Definition 9: Fuzzy Number: A fuzzy number F in a continuous universe U, e.g., a real line, is a fuzzy set F in U which is normal and convex, i.e.,

$$\max_{u \in U} \mu_F(u) = 1, \qquad \text{(normal)}$$

$$\mu_F(\lambda u_1 + (1-\lambda)u_2) \geqslant \min(\mu_F(u_1), \mu_F(u_2)), \qquad \text{(convex)}$$

$$u_1, u_1 \in U, \quad \lambda \in [0,1].$$

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. More specifically: see the following.

Definition 10: Linguistic Variables: A linguistic variable is characterized by a quintuple $(x, T(x), U, G, M)$ in which x is the name of variable; $T(x)$ is the term set of x, that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U; G is a syntactic rule for generating the names of values of x; and M is a semantic rule for associating with each value its meaning. For example, if *speed* is interpreted as a linguistic variable, then its term set T(speed) could be

$$T(\text{speed}) = \{\text{slow, moderate, fast, very slow, more or less fast}, \cdots\}$$

where each term in T(speed) is characterized by a fuzzy set in a universe of discourse $U = [0,100]$. We might interpret "slow" as "a speed below about 40 mph," "moderate" as "a speed close to 55 mph," and "fast" as "a speed above about 70 mph." These terms can be characterized as fuzzy sets whose membership functions are shown in Fig. 1.

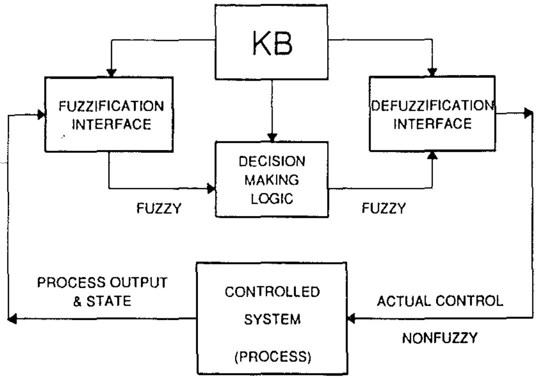


Fig. 2. Basic configuration of fuzzy logic controller (FLC).

D. Fuzzy Logic and Approximate Reasoning

In fuzzy logic and approximate reasoning, there are two important fuzzy implication inference rules named the generalized modus ponens (GMP) and the generalized modus tollens (GMT):

premise 1: x is A',
premise 2: if x is A then y is B,
consequence: y is B' (GMP)

premise 1: y is B′,
premise 2: if x is A then y is B,
consequence: x is A'. (GMT)

The fuzzy implication inference is based on the compositional rule of inference for approximate reasoning suggested by Zadeh in 1973 [146]. Here we introduce fuzzy sets A, A', B, B' via linguistic variables x, y instead of crisp sets in the traditional logic. The GMP, which reduces to "modus ponens" when $A' = A$ and $B' = B$, is closely related to the forward data-driven inference which is particularly useful in the FLC. The GMT, which reduces to "modus tollens" when $B' =$ not B and $A' =$ not A, is closely related to the backward goal-driven inference which is commonly used in expert systems, especially in the realm of medical diagnosis.

Definition 11: Sup-Star Compositional Rule of Inference: If R is a fuzzy relation in $U \times V$, and x is a fuzzy set in U, then the "sup-star compositional rule of inference" asserts that the fuzzy set y in V induced by x is given by [144]

$$y = x \circ R$$

where $x \circ R$ is the sup-star composition of x and R. If the star represents the minimum operator, then this definition reduces to Zadeh's compositional rule of inference [146].

III. MAIN IDEA OF THE FLC

In this section, we present the main ideas underlying the FLC. To highlight the issues involved, Fig. 2 shows the basic configuration of an FLC, which comprises four principal components: a *fuzzification interface*, a knowledge base, *decision-making logic*, and a *defuzzification interface*.

1) The fuzzification interface involves the following functions:
    a) measures the values of input variables,
    b) performs a scale mapping that transfers the range of values of input variables into corresponding universes of discourse,
    c) performs the function of fuzzification that converts input data into suitable linguistic values which may be viewed as labels of fuzzy sets.
2) The knowledge base comprises a knowledge of the application domain and the attendant control goals. It consists of a "data base" and a "linguistic (fuzzy) control rule base:"
    a) the data base provides necessary definitions, which are used to define linguistic control rules and fuzzy data manipulation in an FLC,
    b) the rule base characterizes the control goals and control policy of the domain experts by means of a set of linguistic control rules.
3) The decisionmaking logic is the kernel of an FLC; it has the capability of simulating human decision-making based on fuzzy concepts and of inferring fuzzy control actions employing fuzzy implication and the rules of inference in fuzzy logic.
4) The defuzzification interface performs the following functions:
    a) a scale mapping, which converts the range of values of output variables into corresponding universes of discourse,
    b) defuzzification, which yields a nonfuzzy control action from an inferred fuzzy control action.

We are now ready to describe the main ideas underlying the FLC in terms of fuzzy logic. The structural parameters involved in the design of an FLC will be discussed at a later point.

A. Fuzzy Conditional Statements and Fuzzy Control Rules

In an FLC, the dynamic behavior of a fuzzy system is characterized by a set of linguistic description rules based on expert knowledge. The expert knowledge is usually of the form

IF (a set of conditions are satisfied) THEN (a set of consequences can be inferred).

Since the antecedents and the consequents of these -THEN rules are associated with fuzzy concepts (linguistic rms), they are often called *fuzzy conditional statements*. In our terminology, a *fuzzy control rule* is a fuzzy conditional statement in which the antecedent is a condition in its application domain and the consequent is a control action for the system under control. Basically, fuzzy control rules provide a convenient way for expressing control policy and domain knowledge. Furthermore, several linguistic variables might be involved in the antecedents and the conclusions of these rules. When this is the case, the system will be referred to as a multi-input–multi-output (MIMO) fuzzy system. For example, in the case of two-input–single-output (MISO) fuzzy systems, fuzzy control rules have the form:

$$R_1\text{: if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z \text{ is } C_1,$$
$$R_2\text{: if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z \text{ is } C_2,$$
$$\cdots\ \cdots\ \cdots\ \cdots$$
$$\cdots\ \cdots\ \cdots\ \cdots$$
$$R_n\text{: if } x \text{ is } A_n \text{ and } y \text{ is } B_n \text{ then } z \text{ is } C_n,$$

where x, y, and z are linguistic variables representing two process state variables and one control variable; A_i, B_i, and C_i are linguistic values of the linguistic variables x, y, and z in the universes of discourse U, V, and W, respectively, with $i = 1, 2, \cdots, n$; and an implicit sentence connective also links the rules into a rule set or, equivalently, a rule base.

A fuzzy control rule, such as "if (x is A_i and y is B_i) then (z is C_i)," is implemented by a *fuzzy implication* (fuzzy relation) R_i and is defined as follows:

$$\mu_{R_i} \triangleq \mu_{(A_i \text{ and } B_i \to C_i)}(u,v,w)$$
$$= \left[\mu_{A_i}(u) \text{ and } \mu_{B_i}(v)\right] \to \mu_{C_i}(w)$$

where A_i and B_i is a fuzzy set $A_i \times B_i$ in $U \times V$; $R_i \triangleq (A_i \text{ and } B_i) \to C_i$ is a fuzzy implication (relation) in $U \times V \times W$; and $\to$ denotes a fuzzy implication function. As will be seen later, there are many ways in which a fuzzy implication may be defined.

B. Fuzzification Operator

A fuzzification operator has the effect of transforming crisp data into fuzzy sets. Symbolically,

$$x = \mathit{fuzzifier}(x_0)$$

where x_0 is a crisp input value from a process; x is a fuzzy set; and *fuzzifier* represents a fuzzification operator.

C. Sentence Connective Operators

An FLC consists of a set of fuzzy control rules which are related by the dual concepts of fuzzy implication and the sup-star compositional rule of inference. These fuzzy control rules are combined by using the sentence connectives *and* and *also*. Since each fuzzy control rule is represented by a fuzzy relation, the overall behavior of a fuzzy system is characterized by these fuzzy relations. In other words, a fuzzy system can be characterized by a single fuzzy relation which is the combination of the fuzzy relations in the rule set. The combination in question involves the sentence connective *also*. Symbolically,

$$R = \mathit{also}(R_1, R_2, \cdots, R_i, \cdots, R_n)$$

where *also* represents a sentence connective.

D. Compositional Operator

To infer the output z from the given process states x, y and the fuzzy relation R, the sup-star compositional rule of inference is applied

$$z = y \circ (x \circ R)$$

where $\circ$ is the sup-star composition.

E. Defuzzification Operator

The output of the inference process so far is a fuzzy set, specifying a possibility distribution of control action. In the on-line control, a nonfuzzy (crisp) control action is usually required. Consequently, one must defuzzify the fuzzy control action (output) inferred from the fuzzy control algorithm, namely:

$$z_0 = \mathit{defuzzifier}(z),$$

where z_0 is the nonfuzzy control output and *defuzzifier* is the defuzzification operator.

F. Design Parameters of the FLC

The principal design parameters for an FLC are the following:

1) fuzzification strategies and the interpretation of a fuzzification operator (fuzzifier),
2) data base:
    a) discretization/normalization of universes of discourse,
    b) fuzzy partition of the input and output spaces,
    c) completeness,
    d) choice of the membership function of a primary fuzzy set;
3) rule base:
    a) choice of process state (input) variables and control (output) variables of fuzzy control rules,
    b) source and derivation of fuzzy control rules,
    c) types of fuzzy control rules,
    d) consistency, interactivity, completeness of fuzzy control rules;
4) decision making logic:
    a) definition of a fuzzy implication,
    b) interpretation of the sentence connective *and*,
    c) interpretation of the sentence connective *also*,
    d) definitions of a compositional operator,
    e) inference mechanism;
5) defuzzification strategies and the interpretation of a defuzzification operator (defuzzifier).

IV. FUZZIFICATION STRATEGIES

Fuzzification is related to the vagueness and imprecision in a natural language. It is a subjective valuation which transforms a measurement into a valuation of a subjective value, and hence it could be defined as a mapping from an observed input space to fuzzy sets in certain input universes of discourse. Fuzzification plays an important role in dealing with uncertain information which might be objective or subjective in nature.

In fuzzy control applications, the observed data are usually crisp. Since the data manipulation in an FLC is based on fuzzy set theory, fuzzification is necessary during an earlier stage. Experience with the design of an FLC suggests the following principal ways of dealing with fuzzification.

1) A fuzzification operator "conceptually" converts a crisp value into a fuzzy singleton within a certain universe of discourse. Basically, a fuzzy singleton is a precise value and hence no fuzziness is introduced by fuzzification in this case. This strategy has been widely used in fuzzy control applications since it is natural and easy to implement. It interprets an input x_0 as a fuzzy set A with the membership function $\mu_A(x)$ equal to zero except at the point x_0, at which $\mu_A(x_0)$ equals one.
2) Observed data are disturbed by random noise. In this case, a fuzzification operator should convert the probabilistic data into fuzzy numbers, i.e., fuzzy (possibilistic) data. In this way, computational efficiency is enhanced since fuzzy numbers are much easier to manipulate than random variables. In [76], an isosceles triangle was chosen to be the fuzzification function. The vertex of this triangle corresponds to the mean value of a data set, while the base is twice the standard deviation of the data set. In this way, we form a triangular fuzzy number which is convenient to manipulate [42]. In this connection, it should be noted that Dubois and Prade [20] defined a bijective transformation which transforms a probability measure into a possibility measure by using the concept of the degree of necessity. Basically, the necessity of an event, E, is the added probability of elementary events in E over the probability assigned to the most frequent elementary event outside of E. Based on the method of Dubois and Prade, the histogram of the measured data may be used to estimate the membership function for the transformation of probability into possibility [17].
3) In large scale systems and other applications, some observations relating to the behavior of such systems are precise, while others are measurable only in a statistical sense, and some, referred to as "hybrids," require both probabilistic and possibilistic modes of characterization. The strategy of fuzzification in this case is to use the concept of "hybrid numbers" [42], which involve both uncertainty (fuzzy numbers) and randomness (random numbers). The use of hybrid number arithmetic in the design of an FLC suggests a promising direction that is in need of further exploration.

V. DATA BASE

The knowledge base of an FLC is comprised of two components, namely, a data base and a fuzzy control rule base. We shall address some issues relating to the data

TABLE II
QUANTIZATION AND PRIMARY FUZZY SETS USING A NUMERICAL DEFINITION

Level No.	Range	*NB*	*NM*	*NS*	*ZE*	*PS*	*PM*	*PM*
−6	$x_0 \leqslant -3.2$	1.0	0.3	0.0	0.0	0.0	0.0	0.0
−5	$-3.2 < x_0 \leqslant -1.6$	0.7	0.7	0.0	0.0	0.0	0.0	0.0
−4	$-1.6 < x_0 \leqslant -0.8$	0.3	1.0	0.3	0.0	0.0	0.0	0.0
−3	$-0.8 < x_0 \leqslant -0.4$	0.0	0.7	0.7	0.0	0.0	0.0	0.0
−2	$-0.4 < x_0 \leqslant -0.2$	0.0	0.3	1.0	0.3	0.0	0.0	0.0
−1	$-0.2 < x_0 \leqslant -0.1$	0.0	0.0	0.7	0.7	0.0	0.0	0.0
0	$-0.1 < x_0 \leqslant +0.1$	0.0	0.0	0.3	1.0	0.3	0.0	0.0
1	$+0.1 < x_0 \leqslant +0.2$	0.0	0.0	0.0	0.7	0.7	0.0	0.0
2	$+0.2 < x_0 \leqslant +0.4$	0.0	0.0	0.0	0.3	1.0	0.3	0.0
3	$+0.4 < x_0 \leqslant +0.8$	0.0	0.0	0.0	0.0	0.7	0.7	0.0
4	$+0.8 < x_0 \leqslant +1.6$	0.0	0.0	0.0	0.0	0.3	1.0	0.3
5	$+1.6 < s_0 \leqslant +3.2$	0.0	0.0	0.0	0.0	0.0	0.7	0.7
6	$3.2 \leqslant x_0$	0.0	0.0	0.0	0.0	0.0	0.3	1.0

base in this section and to the rule base in the next section. The concepts associated with a data base are used to characterize fuzzy control rules and fuzzy data manipulation in an FLC. These concepts are subjectively defined and based on experience and engineering judgment. In this connection, it should be noted that the correct choice of the membership functions of a term set plays an essential role in the success of an application. In what follows, we shall discuss some of the important aspects relating to the construction of the data base in an FLC.

A. Discretization / Normalization of Universes of Discourse

The representation of uncertain information with fuzzy sets brings up the problem of quantifying such information for digital computer processing. In general, the representation depends on the nature of the universe of discourse. A universe of discourse in an FLC is either discrete or continuous. If the universe is continuous, a discrete universe may be formed by a discretization of the continuous universe. Furthermore, a continuous universe may be normalized, as will be seen at a later point in this section.

1) Discretization of a Universe of Discourse: Discretization of a universe of discourse is frequently referred to as quantization. In effect, quantization discretizes a universe into a certain number of segments (quantization levels). Each segment is labeled as a generic element, and forms a discrete universe. A fuzzy set is then defined by assigning grade of membership values to each generic element of the new discrete universe. A look-up table based on discrete universes, which defines the output of a controller for all possible combinations of the input signals, can be implemented by off-line processing in order to shorten the running time of the controller [90]. In the case of an FLC with continuous universes, the number of quantization levels should be large enough to provide an adequate approximation and yet be small to save memory storage. The choice of quantization levels has an essential influence on how fine a control can be obtained. For example, if a universe is quantized for every five units of measurement instead of ten units, then the controller is twice as sensitive to the observed variables.

For the purpose of discretization, we need a scale mapping, which serves to transform measured variables into values in the discretized universe. The mapping can be uniform (linear), nonuniform (nonlinear), or both. The choice of quantization levels reflects some *a priori* knowledge. For example, coarse resolution could be used for large errors and fine resolution for small errors. Thus, in a three-input-one-output fuzzy system, we may have control rules of the form:

$$R_i\text{: if error } (e) \text{ is } A_i, \text{ sum of errors } (ie) \text{ is } B_i,$$
$$\text{and change of error } (de) \text{ is } C_i \text{ then output is } D_i.$$

A simple instance of an FLC can be represented by

$$K_4[u(k)] = F[K_1e(k), K_2ie(k), K_3de(k)],$$

where F denotes the fuzzy relation defined by the rule base and $K_i, i = 1,2,3,4$, represents an appropriate scaling mapping. In this relation, we see an analogy to the parameters of a conventional PID controller [63],[105], in which as a special case F is a linear function of its arguments. An example of discretization is shown in Table II, where a universe of discourse is discretized into 13 levels with seven terms (*primary fuzzy sets*) defined on it. In general, due to discretization, the peformance of an FLC is less sensitive to small deviations in the values of the process state variables.

2) Normalization of a Universe of Discourse: The normalization of a universe requires a discretization of the universe of discourse into a finite number of segments, with each segment mapped into a suitable segment of the normalized universe. In this setting a fuzzy set is then defined by assigning an explicit function to its membership function. The normalization of a continuous universe also involves *a priori* knowledge of the input/output space. The scale mapping can be uniform, nonuniform, or both. One example is shown in Table III, where the universe of discourse, $[-6.0, +4.5]$, is transformed into the normalized closed interval $[-1, +1]$.

B. Fuzzy Partition of Input and Output Spaces

A linguistic variable in the antecedent of a fuzzy control rule forms a fuzzy input space with respect to a certain universe of discourse, while that in the consequent

TABLE III
NORMALIZATION AND PRIMARY FUZZY SETS USING A FUNCTIONAL DEFINITION

Normalized Universe	Normalized Segments	Range	u_f	σ_f	Primary Fuzzy Sets
[−1.0, +1.0]	[−1.0, −0.5]	[−6.9, −4.1]	−1.0	0.4	*NB*
	[−0.5, −0.3]	[−4.1, −2.2]	−0.5	0.2	*NM*
	[−0.3, −0.0]	[−2.2, −0.0	−0.2	0.2	*NM*
	[−0.0, +0.2]	[−0.0, +1.0]	0.0	0.2	*ZE*
	[+0.2, +0.6]	[+1.0, +2.5]	0.2	0.2	*PS*
	[+0.6, +1.0]	[+2.5, +4.5]	0.5	0.2	*PM*
			1.0	0.4	*PB*

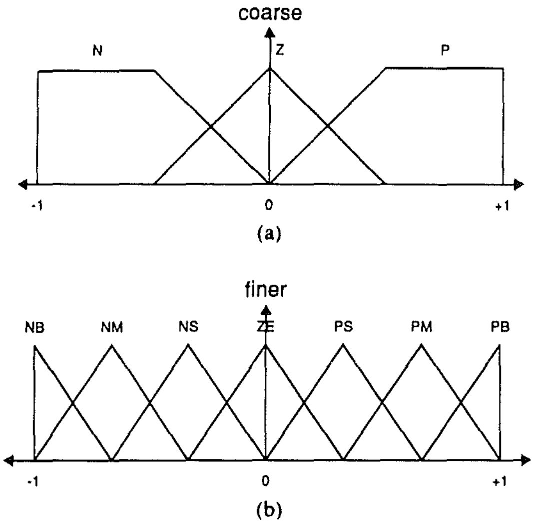


Fig. 3. Diagrammatic representation of fuzzy partition. (a) Coarse fuzzy partition with three terms: *N*, negative; *ZE*, zero; and *P*, positive. (b) Finer fuzzy partition with seven terms: *NB*, negative big; *NM*, negative medium; *NS*, negative small; *ZE*, zero; *PS*, positive small; *PM*, positive medium; and *PB*, positive big.

of the rule forms a fuzzy output space. In general, a linguistic variable is associated with a term set, with each term in the term set defined on the same universe of discourse. A fuzzy partition, then, determines how many terms should exist in a term set. This is equivalent to finding the number of primary fuzzy sets. The number of primary fuzzy sets determines the granularity of the control obtainable with an FLC. The primary fuzzy sets (linguistic terms) usually have a meaning, such as *NB*: negative big; *NM*: negative medium; *NS*: negative small; *ZE*: zero; *PS*: positive small; *PM*: positive medium; and *PB*: positive big. A typical example is shown in Fig. 3, depicting two fuzzy partitions in the same normalized universe $[-1, +1]$. Membership functions having the forms of triangle-shaped and trapezoid-shaped functions are used here. Since a normalized universe implies the knowledge of the input/output space via appropriate scale mappings, a well-formed term set can be achieved as shown. If this is not the case, or a nonnormalized universe is used, the terms could be asymmetrical and unevenly distributed in the universe. Furthermore, the cardinality of a term set in a fuzzy input space determines the maximum number of fuzzy control rules that we can construct. In the case of two-input-one-output fuzzy systems, if the cardinalities of $T(x)$ and $T(y)$ are 3 and 7, respectively, the maximum rule number is 3×7. It should be noted that the fuzzy partition of the fuzzy input/output space is not deterministic and has no unique solution. A heuristic cut and trial procedure is usually needed to find the optimal fuzzy partition.

C. Completeness

Intuitively, a fuzzy control algorithm should always be able to infer a proper control action for every state of process. This property is called "completeness." The completeness of an FLC relates to its data base, rule base, or both.

1) Data Base Strategy: The data base strategy is concerned with the supports on which primary fuzzy sets are defined. The union of these supports should cover the related universe of discourse in relation to some level set ϵ. This property of an FLC is called ϵ-completeness. In general, we choose the level ϵ at the crossover point as shown in Fig. 3, implying that we have a strong belief in the positive sense of the fuzzy control rules which are associated with the FLC. In this sense, a dominant rule always exists and is associated with the degree of belief greater than 0.5. In the extreme case, two dominant rules are activated with equal belief 0.5.

2) Rule Base Strategy: The rule base strategy has to do with the fuzzy control rules themselves. The property of completeness is incorporated into fuzzy control rules through design experience and engineering knowledge. An additional rule is added whenever a fuzzy condition is not included in the rule base, or whenever the degree of partial match between some inputs and the predefined fuzzy conditions is lower than some level, say 0.5. The former shows that no control action will result. The latter indicates that no dominant rule will be fired.

D. Membership Function of a Primary Fuzzy Set

There are two methods used for defining fuzzy sets, depending on whether the universe of discourse is discrete or continuous: a) numerical and b) functional.

1) Numerical Definition: In this case, the grade of membership function of a fuzzy set is represented as a vector of numbers whose dimension depends on the degree of discretization. An illustrative example is shown in Table II. In this case, the membership function of each primary

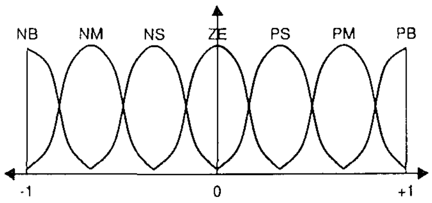


Fig. 4. Example of functional definition of primary fuzzy sets.

fuzzy set has the form of

$$\mu_f(u) = \sum_{i=1}^{5} a_i / u_i,$$

where

$$a = [0.3, 0.7, 1.0, 0.7, 0.3].$$

2) Functional Definition: A functional definition expresses the membership function of a fuzzy set in a functional form, typically a bell-shaped function, triangle-shaped function, trapezoid-shaped function, etc. Such functions are used in FLC because they lead themselves to manipulation through the use of fuzzy arithmetic. The functional definition can readily be adapted to a change in the normalization of a universe. Table III and Fig. 4 show an example of a functional definition expressed as:

$$\mu_f(x) = \exp\left\{ \frac{-(x - u_f)^2}{2\sigma_f^2} \right\}.$$

Note that if the normalized universe is changed, the parameters u_f, σ_f should be changed accordingly.

Either a numerical definition or functional definition may be used to assign the grades of membership to the primary fuzzy sets. The choice of grades of membership is based on the subjective criteria of the decision. In particular, as we mentioned before, if the measurable data might be disturbed by noise, the membership functions should be sufficiently wide to reduce the sensitivity to noise. This raises the issue of the fuzziness or, more accurately, the specificity of a membership function, which affects the robustness of an FLC. This issue is discussed in greater detail in [58].

VI. RULE BASE

A fuzzy system is characterized by a set of linguistic statements based on expert knowledge. The expert knowledge is usually in the form of "if–then" rules, which are easily implemented by fuzzy conditional statements in fuzzy logic. The collection of fuzzy control rules that are expressed as fuzzy conditional statements forms the rule base or the rule set of an FLC. In this section, we shall examine the following topics related to fuzzy control rules: choice of process state (input) variables and control (output) variables, source and derivation, justification, types of fuzzy control rules, and properties of consistency, interactivity, and completeness.

A. Choice of Process State Variables and Control Variables of Fuzzy Control Rules

Fuzzy control-rules are more conveniently formulated in linguistic rather than numerical terms. The proper choice of process state variables and control variables is essential to the characterization of the operation of a fuzzy system. Furthermore, the selection of the linguistic variables has a substantial effect on the performance of an FLC. As was stated earlier, experience and engineering knowledge play an important role during this selection stage. In particular, the choice of linguistic variables and their membership function have a strong influence on the linguistic structure of an FLC. Typically, the linguistic variables in an FLC are the state, state error, state error derivative, state error integral, etc.

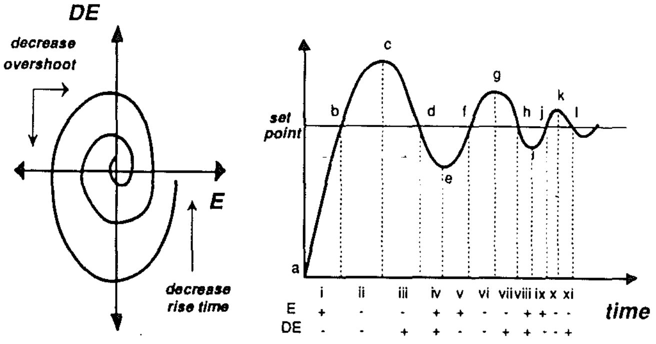
B. Source and Derivation of Fuzzy Control Rules

There are four modes of derivation of fuzzy control rules, as reported in [103]. These four modes are not mutually exclusive, and it seems likely that a combination of them would be necessary to construct an effective method for the derivation of fuzzy control rules.

1) Expert Experience and Control Engineering Knowledge: Fuzzy control rules have the form of fuzzy conditional statements that relate the state variables in the antecedent and process control variables in the consequents. In this connection, it should be noted that in our daily life most of the information on which our decisions are based is linguistic rather than numerical in nature. Seen in this perspective, fuzzy control rules provide a natural framework for the characterization of human behavior and decisions analysis. Many experts have found that fuzzy control rules provide a convenient way to express their domain knowledge. This explains why most FLCs are based on the knowledge and experience which are expressed in the language of fuzzy if–then rules [64], [47], [50], [80], [82], [59], [118], [113], [58], [127], [4].

The formulation of fuzzy control rules can be achieved by means of two heuristic approaches. The most common one involves an introspective verbalization of human expertise. A typical example of such verbalization is the operating manual for a cement kiln. Another approach includes an interrogation of experienced experts or operators using a carefully organized questionnaire. In this manner, we can form a prototype of fuzzy control rules for a practicular application domain. For optimized performance, the use of cut and trial procedures is usually a necessity.

2) Based on Operator's Control Actions: In many industrial man–machine control systems, the input–output relations are not known with sufficient precision to make it possible to employ classical control theory for modeling and simulation. And yet skilled human operators can control such systems quite successfully without having any quantitative models in mind. In effect, a human operator employs–consciously or subconsciously—a set of fuzzy



(a) (b)

Fig. 5. Rule justification by using phase plane. (a) Phase–plane trajectory. (b) System step response.

if–then rules to control the process. As was pointed out by Sugeno, to automate such processes, it is expedient to express the operator's control rules as fuzzy if–then rules employing linguistic variables. In practice, such rules can be deduced from the observation of human controller's actions in terms of the input–output operating data [97]–[99].

3) Based on the Fuzzy Model of a Process: In the linguistic approach, the linguistic description of the dynamic characteristics of a controlled process may be viewed as a fuzzy model of the process. Based on the fuzzy model, we can generate a set of fuzzy control rules for attaining optimal performance of a dynamic system. The set of fuzzy control rules forms the rule base of an FLC. Although this approach is somewhat more complicated, it yields better performance and reliability, and provides a more tractable structure for dealing theoretically with the FLC. However, this approach to the design of an FLC has not as yet been fully developed.

4) Based on Learning: Many FLCs have been built to emulate human decision-making behavior, but few are focused on human learning, namely, the ability to create fuzzy control rules and to modify them based on experience. Procyk and Mamdani [87] described the first self-organizing controller (SOC). The SOC has a hierarchical structure which consists of two rule bases. The first one is the general rule base of an FLC. The second one is constructed by "meta-rules" which exhibit human-like learning ability to create and modify the general rule base based on the desired overall performance of the system. Recently, further studies relating to the SOC have been carried out at Queen Mary College and elsewhere [60], [94], [102], [95], [106]. A very interesting example of a fuzzy rule-based system which has a learning capability is Sugeno's fuzzy car [97], [99]. Sugeno's fuzzy car can be trained to park by itself.

C. Justification of Fuzzy Control Rules

There are two principal approaches to the derivation of fuzzy control rules. The first is a heuristic method in which a collection of fuzzy control rules is formed by analyzing the behavior of a controlled process. The control rules are derived in such a way that the deviation from a desired state can be corrected and the control objective can be achieved. The derivation is purely heuristic in nature and relies on the qualitative knowledge of process behavior. Several methods of adjustment of rule selection have been studied [1], [49], [7], [6]. A brief review of these results is given in the following. The second approach is basically a deterministic method which can systematically determine the linguistic structure and/or parameters of the fuzzy control rules that satisfy the control objectives and constraints [111], [103], [104], [101].

Mamdani [1] proposed a prescriptive algorithm for deriving the "best" control rules by restricting system responses to a "prescriptive fuzzy band" which is specified by fuzzy control rules. However, the convergence of the prescriptive method requires a careful analysis.

King and Mamdani [49] introduced another useful method for rule justification. So-called "scale mappings" should be adjusted first so that the system trajectory can terminate on a desired state. The rule justification is done by referring to a closed system trajectory in a phase plane. A knowledge of parameter-adjusting based on phase plane analysis (e.g., overshoot, rise time) and an intuitive feel for the behavior of the closed loop system are required. The principle of global rule modification in symmetry and monotonicity is also employed.

For example, Fig. 5 shows the system response of a process to be controlled, where the input variables of the FLC are the error (E) and error derivative (DE). The output is the change of the process input (CI). We assume that the term sets of input/output variables have the same cardinality, 3, with a common term {negative, zero, positive}. The prototype of fuzzy control rules is tabulated in Table IV and a justification of fuzzy control rules is added in Table V. The corresponding rule of region i can be formulated as rule R_i and has the effect of shortening the rise time. Rule R_{ii} for region ii decreases the overshoot of the system's response. More specifically,

R_i: if (E is positive and DE is negative)
then CI is positive,

R_{ii}: if (E is negative and DE is negative)
then CI is negative.

TABLE IV
PROTOTYPE OF FUZZY CONTROL RULES WITH TERM SETS {NEGATIVE, ZERO, POSITIVE}

Rule No.	*E*	*DE*	*CI*	Reference Point
1	*P*	*Z*	*P*	a, e, i
2	*Z*	*N*	*N*	b, f, j
3	*N*	*Z*	*N*	c, g, k
4	*Z*	*P*	*P*	d, h, l
5	*Z*	*Z*	*Z*	set point

TABLE V
RULE JUSTIFICATION WITH TERM SETS {NEGATIVE, ZERO, POSITIVE}

Rule No.	*E*	*DE*	*CI*	Reference Range
6	*P*	*N*	*P*	i (rise time), v
7	*N*	*N*	*N*	ii (overshoot), vi
8	*N*	*P*	*N*	iii, vii
9	*P*	*P*	*P*	iv, viii
10	*P*	*N*	*Z*	ix
11	*N*	*P*	*Z*	xi

TABLE VI
PROTOTYPE OF FUZZY CONTROL RULES WITH TERM SETS {*NB, NM, NS, ZE, PS, PM, PB*}

Rule No.	*E*	*DE*	*CI*	Reference Point
1	*PB*	*ZE*	*PB*	a
2	*PM*	*ZE*	*PM*	e
3	*PS*	*ZE*	*PS*	i
4	*ZE*	*NB*	*NB*	b
5	*ZE*	*NM*	*NM*	f
6	*ZE*	*NS*	*NS*	j
7	*NB*	*ZE*	*NB*	c
8	*NM*	*ZE*	*NM*	g
9	*NS*	*ZE*	*NS*	k
10	*ZE*	*PB*	*PB*	d
11	*ZE*	*PM*	*PM*	h
12	*ZE*	*PS*	*PS*	l
13	*ZE*	*ZE*	*ZE*	set point

TABLE VII
RULE JUSTIFICATION WITH TERM SETS {*NB, NM, NS, ZE, PS, PM, PM*}

Rule No.	*E*	*DE*	*CI*	Reference Range
14	*PB*	*NS*	*PM*	i (rise time)
15	*PS*	*NB*	*NM*	i (overshoot)
16	*NB*	*PS*	*NM*	iii
17	*NS*	*PB*	*PM*	iii
18	*PS*	*NS*	*ZE*	ix
19	*NS*	*PS*	*ZE*	xi

Better control performance can be obtained by using finer fuzzy partitioned subspaces, for example, with the term set {*NB, NM, NS, ZE, PS, PM, PB*}. The prototype and the justification of fuzzy control rules are also given in Table VI and Table VII.

A slightly modified method was suggested in [7]. It tracked the linguistic trajectory of a closed loop system in "linguistic phase plane." The main idea is that scale mappings should be adjusted first to yield approximately a desired trajectory behavior. This can be inferred from the linguistic trajectories. Then rule modification can be accomplished by using the linguistic trajectory behavior to optimize the system response in the linguistic phase plane.

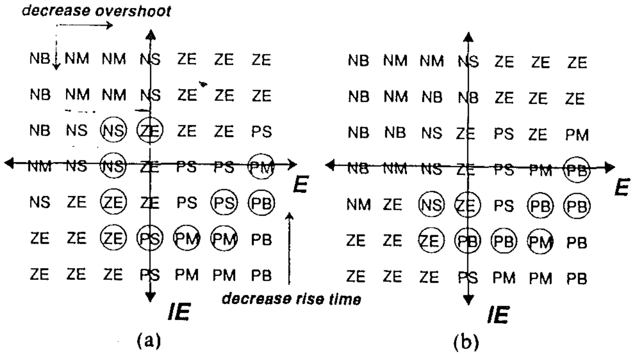


Fig. 6. Rule justification by using a linguistic phase plane. (a) Linguistic trajectory with initial rules. (b) Linguistic trajectory with modified rules. (From Braae and Rutherford [7].

An additional advantage of this approach is that the measurement noise appearing in the linguistic phase plane is less of a problem than that in the nonlinguistic phase plane. An example is shown in Fig. 6.

An approach to generating the rule base of an FLC, which is analogous to the conventional controller design by pole placement, is described in [6]. Braae and Rutherford assumed that the fuzzy control rules of an open system (process) and a desired closed-loop system were initially given. The purpose is to synthesize a linguistic control element (FLC) based on the fuzzy models described above. The main idea is to invert the low order linguistic model of a certain open loop system. However, linguistic inversion mappings are usually incomplete or multivalued. So, an "approximate" strategy, which is somewhat heuristic and subjective, is necessary to complete the inverse mapping which has a reasonable singled-valued solution. This approximation has substantial effect on "linguistic substitution" which further determines a fuzzy controller. This method is restricted to relative low order systems but it provides an explicit solution for rule generation of the FLC, assuming that fuzzy models of the open and closed systems are available.

The systematic rule justification has recently been proposed and studied by means of fuzzy relational equations [13], [15], [84], [125] and linguistic control rules [111], [103], [104], [101]. The basic notion of these two approaches is so-called "fuzzy identification." As in conventional identification, the fuzzy identification comprises two phases, namely, structure identification and parameter estimation. The studies in question deal with one, or both.

Tong [111] introduced the concept of "logical examination" (LE) for converting process input–output data into a set of fuzzy control rules. Tong tackled both identification problems simultaneously, and used a correlation analysis of the LE to determine the linguistic structure. However, it is still somewhat heuristic and subjective, and encounters difficulties in the identification of multivariable fuzzy systems.

Takagi and Sugeno [103] proposed a fuzzy identification algorithm for modeling human operator's control actions. In this case, a suitable linguistic structure is easy to find since one can observe and/or ask for the kind of information which the operator needs, such as process state variables. The fuzzy control rules to be identified have the form of

$$R_i\text{: if } x \text{ is } A_i, \cdots, \text{ and } y \text{ is } B_i \text{ then } z = f_i(x, \cdots, y)$$

where $x, \cdots, y$, and z are linguistic variables representing the process state variables and the control variable; $A_i, \cdots, B_i$ are linguistic terms of the linguistic variables $x, \cdots, y$, and z in the universes of discourse $U, \cdots, V$, and W, respectively, with $i = 1, 2, \cdots, n$; and z is a logical function of the process state variables such as a linear function of $x, \cdots, y$. In this way, the problem is reduced to parameter estimation, which is done by optimizing a least-square performance index via a weighted linear regression method. The inference mechanism of this FLC will be discussed in Part II [150].

Sugeno has successfully applied this method to the design of an FLC for navigating a model car through a crank-shaped curve [98] and for parking a model car in a garage [97],[99]. Sugeno's method provides a more systematic approach to the design of an FLC, and the experimental results are quite remarkable. However, some steps of this algorithm, such as the choice of process state variables, the fuzzy partition of input spaces, and the choice of the membership functions of primary fuzzy sets, depend on trial-and-error.

Recently, Takagi and Sugeno [104] improved their algorithm so that parameter estimation can be fully implemented. At issue is the problem of structure identification, which is partly addressed in this paper. Further research on this problem has been reported by Sugeno and Kang in [101].

Another approach based on fuzzy relational equations is directed at the same problems. The structure identification requires the determination of the system order and time delays of discrete-time fuzzy models, while the parameter estimation reduces to the determination of the overall fuzzy relation matrix from the input-output data of the system. The reader is referred to [13],[15],[84],[125] for further details.

D. Types of Fuzzy Control Rules

Depending on their nature, two types of fuzzy control rules, state evaluation fuzzy control rules and object evaluation fuzzy control rules, are currently in use in the design of the FLC.

1) State Evaluation Fuzzy Control Rules: Most FLC's have state evaluation fuzzy control rules which, in the case of MISO systems, are characterized as a collection of rules of the form

$$R_1\text{: if } x \text{ is } A_1, \cdots, \text{ and } y \text{ is } B_1 \text{ then } z \text{ is } C_1$$
$$R_2\text{: if } x \text{ is } A_2, \cdots, \text{ and } y \text{ is } B_2 \text{ then } z \text{ is } C_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$R_n\text{: if } x \text{ is } A_n, \cdots, \text{ and } y \text{ is } B_n \text{ then } z \text{ is } C_n$$

where $x, \cdots, y$, and z are linguistic variables representing the process state variables and the control variable; $A_i, \cdots, B_i$, and C_i are the linguistic values of the linguistic variables $x, \cdots, y$, and z in the universes of discourse $U, \cdots, V$, and W, respectively, $i = 1, 2, \cdots, n$.

In a more general version, the consequent is represented as a function of the process state variables $x, \cdots, y$, i.e.,

$$R_i\text{: if } x \text{ is } A_i, \cdots, \text{ and } y \text{ is } B_i \text{ then } z = f_i(x, \cdots, y).$$

Fuzzy control rules of this type, which are referred to as "state evaluation fuzzy control rules," evaluate the process state (e.g., state, state error, state integral) at time t and compute a fuzzy control action at time t as a function of $(x, \cdots, y)$ and the control rules in the rule set.

2) Object Evaluation Fuzzy Control Rules: Yasunobu, Miyamoto, and Ihara [135] proposed another algorithm which predicts present and future control actions and evaluates control objectives. It is called "object evaluation fuzzy control," or "predictive fuzzy control." The rules in question, which are derived from skilled operator's experience, are referred to as "object evaluation fuzzy control rules." A typical rule is described as

$$R_i\text{: if } (u \text{ is } C_i \rightarrow (x \text{ is } A_i \text{ and } y \text{ is } B_i)) \text{ then } u \text{ is } C_i.$$

A control command is inferred from an objective evaluation of a fuzzy control result that satisfies the desired states and objectives. A control command u takes a crisp set as a value, and x, y are performance indices for the evaluation of the ith rule, taking values such as "good" or "bad." The most likely control rule is selected through predicting the results (x, y) corresponding to every control command C_i.

In linguistic terms, the rule is interpreted as: "if the performance index x is A_i and index y is B_i when a control command u is chosen to be C_i, then this rule is selected and the control command C_i is taken to be the output of the controller."

In automatic train operation, a typical control rule is *if the control notch is not changed and if the train stops in the predetermined allowance zone, then the control notch is not changed.*

It is well known that systems control encounters difficulties in satisfying multiple performance indices simultaneously and in achieving accurate control in the presence of disturbances. In such circumstances, fuzzy control provides an effective framework for solution. However, the state evaluation fuzzy control does not evaluate the computed control actions as human operators do. By contrast, the predictive fuzzy control provides a mechanism for evaluation so that the desired states and control objectives can be achieved more easily. It should be noted that predictive control has been successfully applied to automatic train operation [135], [136], [139] as well as to automatic container crane operation systems [137]–[139]. Tests have shown that this type of control is capable of operating trains and cranes as skillfully as an experienced operator.

E. Properties of Consistency, Interactivity, and Completeness

1) Completeness: Please refer to Section V of this paper.

2) Number of Fuzzy Control Rules: There is no general procedure for deciding on the optimal number of fuzzy control rules since a number of factors are involved in the decision, e.g., performance of the controller, efficiency of computation, human operator behavior, and the choice of linguistic variables.

3) Consistency of Fuzzy Control Rules: If the derivation of fuzzy control rules is based on the human operator experience, the rules may be subjected to different performance criteria. In practice, it is important to check on the consistency of fuzzy control rules in order to minimize the possibility of contradiction. [64],[12].

4) Interactivity of Fuzzy Control Rules: Assuming that a collection of fuzzy control rules has the form

$$R_i\text{: if } x \text{ is } A_i \text{ then } z \text{ is } C_i, \qquad i = 1, \cdots, n.$$

If an input x_0 is A_i, we would expect that the control action z is C_i. In fact, the control action z may be a subset or a superset of C_i [12], [26], [85], [18], [19], depending on the definition of fuzzy implication and sup-star composition. This may happen as a consequence of interaction between the rules.

The problem of interaction is complex and not as yet well understood. The reported research in [12], [26], [85], [18], [19] indicates that interactivity of rules can be controlled by the choice of fuzzy implication and sup-star composition. The consistency of rules may be improved through the use of the concept of a fuzzy clustering of fuzzy control rules. In this connection, it should be noted that Sugeno's reasoning and identification algorithm provides an alternative solution to these problems [104], [101].

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REFERENCES

[1] N. Baaklini and E. H. Mamdani, "Prescriptive methods for deriving control policy in a fuzzy-logic controller," *Electron. Lett.*, vol. 11, pp. 625–626, 1975.

[2] J. Baldwin and N. Guild, "Modeling controllers using fuzzy relations," *Kybernetes*, vol. 9, pp. 223–229, 1980.

[3] J. F. Baldwin and B. W. Pilsworth, "Axiomatic approach to implication for approximate reasoning with fuzzy logic," *Fuzzy Sets Syst.*, vol 3, pp. 193–219, 1980.

[4] J. A. Bernard, "Use of rule-based system for process control," *IEEE Contr. Syst. Mag.*, vol. 8, no. 5, pp. 3–13, 1988.

[5] M. Braae and D. A. Rutherford, "Fuzzy relations in a control setting," *Kybernetes*, vol. 7, no. 3, pp. 185–188, 1978.

[6] ____, "Theoretical and linguistic aspects of the fuzzy logic controller," *Automat.*, vol. 15, no 5, pp. 553–577, 1979.

[7] ____, "Section of parameters for a fuzzy logic controller," *Fuzzy Sets Syst.*, vol. 2, no. 3, pp. 185–199, 1979.

[8] K. Ciliz, J. Fei, K. Usluel, and C. Isik, "Pratical aspects of the knowledge-based control of a mobile robot motion," in *Proc. 30th Midwest Symp. on Circuits and Systems*, Syracuse, NY, 1987.

[9] E. Czogala, *Probabilistic Sets in Decision Making and Control.* Cologne: Verlag TUV Rheinland, 1984.

[10] ____, "Probabilistic fuzzy controller as a generalization of the concept of fuzzy controller," *Fuzzy Sets Syst.*, vol. 26, pp. 215–223, 1988.

[11] E. Czogala and K. Hirota, *Probabilistic Sets: Fuzzy and Stochastic Approach to Decision, Control and Recognition Processes.* Cologne: Verlag TUV Rheinland, 1986.

[12] E. Czogala and W. Pedrycz, "Some problems concerning the construction of algorithms of decision-making in fuzzy systems," *Int. J. Man. Mach. Studies*, vol. 15, pp. 201–221, 1981.

[13] ____, "On identification in fuzzy systems and its applications in control problems," *Fuzzy Sets Syst.*, vol. 6, no. 1, pp. 73–83, 1981.

[14] ____, "Control problems in fuzzy systems," *Fuzzy Sets Syst.*, vol. 7, no. 3, pp. 257–274, 1982.

[15] ____, , "Fuzzy rule generation for fuzzy control," *Cybern. Syst.*, vol. 13, no. 3, pp. 275–293, 1982.

[16] E. Czogala and T. Rawlik, "Modelling of a fuzzy controller with application to the control of biological processes," *Fuzzy Sets Syst.*, vol. 31, pp. 13–22, 1989.

[17] B. Bharathi Devi and V. V. S. Sarma, "Estimation of fuzzy memberships from histograms," *Inform. Sci.*, vol. 35, pp. 43–59, 1985.

[18] D. Dubois and H. Prade, "Fuzzy logics and the generalized modus ponens revisited," *Cybern. Syst.*, vol. 15, pp. 3–4, 1984.

[19] ____, "The generalized modus ponens under sup-min composition—A theoretical study." in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Eds. Amsterdam: North-Holland, 1985, pp. 217–232.

[20] ____, "Unfair coins and necessity measures: Toward a possibilistic interpretation of histograms," *Fuzzy Sets Syst.*, vol. 10, no. 1, pp. 15–20, 1985.

[21] ____, *Possibility Theory: An Approach to Computerized Processing of Uncertainty.* New York: Plenum Press, 1988.

[22] J. Efstathiou, "Rule-based process control using fuzzy logic," in *Approximate Reasoning in Intelligent Systems, Decision and Control*, E. Sanchez and L. A. Zadeh, Eds. New York: Pergamon, 1987, pp. 145–158.

[23] F. Fujitec, "FLEX-8800 series elevator group control system," Fujitec Co., Ltd., Osaka, Japan, 1988.

[24] S. Fukami, M. Mizumoto, and K. Tanaka, "Some considerations of fuzzy conditional inference," *Fuzzy Sets Syst.*, vol. 4, pp. 243–273, 1980.

[25] B. R. Gaines and L. J. Kohout, "The fuzzy decade: A bibliography of fuzzy systems and closely related topics," *Int. J. Man. Mach. Studies*, vol. 9, pp. 1–68, 1977.

[26] S. Gottwald and W. Pedrycz, "Problems of the design of fuzzy controllers," in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Ed. Amsterdam: North-Holland, 1985, pp. 393–405.

[27] B. P. Graham and R. B. Newell, "Fuzzy identification and control of a liquid level rig," *Fuzzy Sets Syst.*, vol. 26, pp. 255–273, 1988.

[28] ____, "Fuzzy adaptive control of a first-order process," *Fuzzy Sets Syst.*, vol. 31, pp. 47–65, 1989.

[29] I. Graham and P. L. Jones, *Expert Systems Knowledge, Uncertainty and Decision.* New York: Chaman and Hall, 1988.

[30] M. M. Gupta and Y. Tsukamoto, "Fuzzy logic controllers—A perspective," in *Proc. Joint Automatic Control Conf.*, San Francisco, Aug. 1980, pp. FA10-C.

[31] M. M. Gupta, G. M. Trojan, and J. B. Kiszka, "Controllability of fuzzy control systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-16, no 4, pp. 576–582, 1986.

[32] K. Hirota, Y. Arai, and S. Hachisu, "Moving mark recognition and moving object manipulation in fuzzy controlled robot," *Cont. Theory Adv. Technol.*, vol. 2, no. 3, pp. 399–418, 1986.

[33] K. Hirota and W. Pedrycz, "Analysis and synthesis of fuzzy systems by the use of fuzzy sets," *Fuzzy Sets Syst.*, vol. 10, no 1, pp. 1–14, 1983.

[34] C. Isik, "Identification and fuzzy rule-based control of a mobile robot motion," in *Proc. IEEE Int. Symposium Intelligent Control*, Philadelphia, PA, 1987.

[35] O. Itoh, K. Gotoh, T. Nakayama, and S. Takamizawa, "Application of fuzzy control to activated sludge process," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July, 1987, pp. 282–285.

[36] A. Jones, A Kaufmann, and H. J. Zimmermann, *Fuzzy Sets—Theory and Applications*. Boston: D. Reidel Publishing Company, 1986.

[37] J. Kacprzyk and S. A. Orlovski, *Optimization Models using Fuzzy Sets and Possibility Theory*. Boston: D. Reidel Publishing Company, 1987.

[38] J. Kacprzyk and M. Fedrizzi, *Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making*. New York: Springer-Verlag, 1988.

[39] A. Kandel, *Fuzzy Techniques in Pattern Recognition*. New York: Wiley-Interscience, 1982.

[40] Y. Kasai and Y. Morimoto, "Electronically controlled continuously variable transmission," in *Proc. Int. Congress on Transportation Electronics*, Dearborn, MI, 1988.

[41] A. Kaufmann, *Introduction to Theory of Fuzzy Subsets*. New York: Academic, 1975.

[42] A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic*. New York: Van Nostrand, 1985.

[43] ____, *Fuzzy Mathematical Models in Engineering and Management Science*. Amsterdam: North-Holland, 1988.

[44] M. Kelemen, Fuzziness and structural stability," *Kybernetes*, vol. 10, pp. 67–69, 1981.

[45] W. J. M. Kickert, "Towards an analysis of linguistic modeling," *Fuzzy Sets Syst.*, vol. 2, pp. 293–307, 1979.

[46] ____, "Further analysis and application of fuzzy logic control." Internal Rep. F/WK2/75, Queen Mary College, London, 1975.

[47] W. J. M. Kickert and H. R. Van Nauta Lemke, "Application of a fuzzy controller in a warm water plant," *Automat.*, vol. 12, no. 4, pp. 301–308, 1976.

[48] W. J. M. Kickert and E. H. Mamdani, "Analysis of a fuzzy logic controller," *Fuzzy Sets Syst.*, vol. 1, no. 1, pp. 29–44, 1978.

[49] P. J. King and E. H. Mamdani, "The application of fuzzy control systems to industrial processes," in *IFAC World Congress*, MIT, Boston, 1975.

[50] ____, "The application of fuzzy control systems to industrial processes," *Automat.*, vol. 13, no. 3, pp. 235–242, 1977.

[51] M. Kinoshita, T. Fukuzaki, T. Satoh, and M. Miyake, "An automatic operation method for control rods in BWR plants," in *Proc. Specialists' Meeting on In-Core Instrumentation and Reactor Core Assessment*, Cadarache, France, 1988.

[52] J. B. Kiszka, M. E. Kochanska, and D. S. Sliwinska, "The influence of some parameters on the accuracy of fuzzy model," in *Industrial Applications of Fuzzy Control*. M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 187–230.

[53] ____, "The influence of some fuzzy implication operators on the accuracy of a fuzzy model—Part I," *Fuzzy Sets Syst.*, vol. 15, pp. 111–128, 1985.

[54] ____, "The influence of some fuzzy implication operators on the accuracy of a fuzzy model—Part II." *Fuzzy Sets Syst.*, vol. 15, pp. 223–240, 1985.

[55] J. B. Kiszka, M. M. Gupta, and P. N. Nikiforuk, "Energetistic stability of fuzzy dynamic systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 5, pp. 783–792, 1985.

[56] G. J. Klir and T. A. Folger, *Fuzzy Sets, Uncertainty, and Information*. Englewood Cliffs, NJ: Prentice Hall, 1988.

[57] S. V. Komolov, S. P. Makeev, and I. F. Shaknov, "Optimal control of a finite automation with fuzzy constraints and a fuzzy target," *Cybern.*, vol. 16, no. 6, pp. 805–810, 1979.

[58] L. I. Larkin, "A fuzzy logic controller for aircraft flight control," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 87–104.

[59] P. M. Larsen, "Industrial applications of fuzzy logic control," *Int. J. Man Mach. Studies*, vol. 12, no. 1, pp. 3–10, 1980.

[60] E. Lembessis, "Dynamical learning behavior of a rule-based self-organising controller," Ph.D. thesis, Queen Mary College, Univ. of London, 1984.

[61] Y. F. Li and C. C. Lau, "Development of fuzzy algorithms for servo systems," *IEEE Contr. Syst. Mag.*, vol. 9, no. 3, pp. 65–72, 1989.

[62] J. Maiers and Y. S. Sherif, "Applications of fuzzy set theory," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 1, pp. 175–189, 1985.

[63] E. H. Mamdani, "Applications of fuzzy algorithms for simple dynamic plant," *Proc. IEE*, vol. 121, no. 12, pp. 1585–1588, 1974.

[64] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *Int. J. Man Mach. Studies*, vol. 7, no 1, pp. 1–13, 1975.

[65] E. H. Mamdani, "Advances in the linguistic synthesis of fuzzy controllers," *Int. J. Man Mach. Studies*, vol. 8, no. 6, pp. 669–678, 1976.

[66] ____, "Application of fuzzy logic to approximate reasoning using linguistic synthesis," *IEEE Trans. Computer*, vol. C-26, no. 12, pp. 1182–1191, 1977.

[67] ____, "Applications of fuzzy set theory to control systems: a survey," in *Fuzzy Automata and Decision Processes*, M. M. Gupta, G. N. Sardis, and B. R. Gaines, Eds. Amsterdam: North-Holland, 1977, pp. 77–88.

[68] E. H. Mamdani and B. R. Gaines, *Fuzzy Reasoning and its Applications*. London: Academic, 1981.

[69] K. Matsushima and H. Sugiyama, "Human operator's fuzzy model in man-machine system with a nonlinear control object," in *First Int. Conf. on Fuzzy Information Processing*, 1984.

[70] M. Mizumoto, "Note on the arithmetic rule by Zadeh for fuzzy conditional inference," *Cybern. Syst.*, vol. 12, pp. 247–306, 1981.

[71] M. Mizumoto and H. Zimmermann, "Comparison of fuzzy reasoning methods," *Fuzzy Sets and Systems*, vol. 8, pp. 253-283, 1982.

[72] M. Mizumoto, "Comparison of various fuzzy reasoning methods," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 2–7.

[73] ____, "Fuzzy controls under various approximate reasoning methods," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July, 1987, pp. 143–146.

[74] S. Murakami, "Application of fuzzy controller to automobile speed control system," in *Proc. of the IFAC Symp. on Fuzzy Information, Knowledge Representation and Decision Analysis*, Marseille, France, 1983, pp. 43–48.

[75] S. Murakami and M. Maeda, "Application of fuzzy controller to automobile speed control system," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 105–124.

[76] Y. Murayama and T. Terano, "Optimizing control of diesel engine," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 63–72.

[77] C. V. Negoita, "On the stability of fuzzy systems," in *Proc. IEEE Int. Conf. on Cybernetics and Society*, 1978, pp. 936–937.

[78] ____, *Expert Systems and Fuzzy Systems*. Menlo Park: Benjamin/Cummings Publishing Company, 1985.

[79] A. Di Nola and A. G. S. Ventre, *The Mathematics of Fuzzy Systems*. Gesamtherstellung: Verlag TUV Rheinland, 1986.

[80] J. J. Ostergaad, "Fuzzy logic control of a heat exchange process," in *Fuzzy Automata and Decision Processes*, M. M. Gupta, G. N. Saridis, and B. R. Gaines, Eds. Amsterdam: North-Holland, 1977, pp. 285–320.

[81] S. K. Pal and D. K. Dutta Majumder, *Fuzzy Mathematical Approach to Pattern Recognition*. New Delhi: Wiley Eastern Limited, 1986.

[82] C. P. Pappis and E. H. Mamdani, "A fuzzy logic controller for a traffic junction," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-7, no. 10, pp. 707–717, 1977.

[83] W. Pedrycz, "An approach to the analysis of fuzzy systems," *Int. J. Control*, vol. 34, no. 3, pp. 403–421, 1981.

[84] ____, "An identification of fuzzy relational systems," *Fuzzy Sets Syst.*, vol. 13, pp. 153–167, 1984.

[85] ____, "Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data," *Fuzzy Sets Syst.*, vol. 16, pp. 163–175, 1985.

[86] H. Prade and C. V. Negoita, *Fuzzy Logic in Knowledge Engineering*. Gesamtherstellung: Verlag TUV Rheinland, 1986.

[87] T. J. Procyk and E. H. Mamdani, "A linguistic self-organizing process controller," *Automat.*, vol. 15, no. 1, pp. 15–30, 1979.

[88] G. P. Rao and D. A. Rutherford, "Approximate reconstruction of mapping functions from linguistic description in problems of fuzzy logic applied to system control," *Cybernet.*, vol. 12, no. 3, pp. 225–236, 1981.

[89] K. S. Ray and D. D. Majumder, "Application of circle criteria for stability analysis of linear SISO and MIMO systems associated with fuzzy logic controller," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-14, no. 2, pp. 345–349, 1984.

[90] D. A. Rutherford and G. C. Bloore, "The implementation of fuzzy algorithms for control," in *Proc. IEEE*, vol. 64, no. 4, pp.

572–573, 1976.

[91] D. A. Rutherford and G. A. Carter, "A Heuristic adaptive controller for a sinter plant," in *Proc. 2nd IFAC Symp. on Automation in Mining, Mineral and Metal Processing*, Hohannesburg, RSA, Sept. 1976.

[92] Y. Sakai, "A fuzzy controller in turning process automation," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 139–152.

[93] E. Sanchez and L. A. Zadeh, *Approximate Reasoning in Intelligent Systems, Decision and Control.* New York: Pergamnon, 1988.

[94] E. M. Scharf and N. J. Mandic, "The application of a fuzzy controller to the control of a multi-degree-freedom robot arm," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 41–62.

[95] S. Shao, "Fuzzy self-organizing controller and its application for dynamic processes," *Fuzzy Sets Syst.*, vol. 26, pp. 151–164, 1988.

[96] M. S. Stachowicz and M. E. Kochanska, "Fuzzy modeling of the process," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 86–89.

[97] M. Sugeno and K. Murakami, "Fuzzy parking control of model car," in *23rd IEEE Conf. on Decision and Control*, Las Vegas, 1984.

[98] M. Sugeno and M. Nishida, "Fuzzy control of model car," *Fuzzy Sets Syst.*, vol. 16, pp. 103–113, 1985.

[99] M. Sugeno and K. Murakami, "An experimental study on fuzzy parking control using a model car," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 125–138.

[100] M. Sugeno, "An introductory survey of fuzzy control," *Inform. Sci.*, vol. 36, pp. 59–83, 1985.

[101] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets Syst.*, vol. 28, no. 1, pp. 15–33, 1988.

[102] K. Sugiyama, "Analysis and synthesis of the rule based self-organising controller," Ph.D thesis, Queen Mary College, Univ. of London, 1986.

[103] T. Takagi and M. Sugeno, "Derivation of fuzzy control rules from human operator's control actions," in *Proc. of the IFAC Symp. on Fuzzy Information, Knowledge Representation and Decision Analysis*, Marseilles, France, July 1983, pp. 55–60.

[104] ____, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 1, pp. 116–132, 1985.

[105] K. L. Tang and R. J. Mulholland, "Comparing fuzzy logic with classical controller designs," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-17, no. 6, pp. 1085–1087, 1987.

[106] R. Tanscheit and E. M. Scharf, "Experiments with the use of a rule-based self-organising controller for robotics applications," *Fuzzy Sets Syst.*, vol. 26, pp. 195–214, 1988.

[107] M. Togai and H. Watanabe, "Expert system on a chip: An engine for real-time approximate reasoning," *IEEE Expert Syst. Mag.*, vol. 1, pp. 55–62, 1986.

[108] M. Togai and S. Chiu, "A fuzzy accelerator and a programming environment for real-time fuzzy control," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 147–151.

[109] R. M. Tong, "Analysis of fuzzy control algorithms using the relation matrix," *Int. J. Man Mach. Studies*, vol. 8, no. 6, pp. 679–686, 1976.

[110] ____, "A control engineering review of fuzzy systems," *Automa.*, vol. 13, no. 6, pp. 559–569, 1977.

[111] ____, "Synthesis of fuzzy models for industrial processes," *Int. Gen. Syst.*, vol. 4, pp. 143–162, 1978.

[112] ____, "The evaluation of fuzzy models derived from experimental data," *Fuzzy Sets Syst.*, vol. 4, pp. 1–12, 1980.

[113] R. M. Tong, M. B. Beck, and A. Latten, "Fuzzy control of the activated sludge wastewater treatment process," *Automat.*, vol. 16, no. 6, pp. 695–701, 1980.

[114] R. M. Tong, "A retrospective view of fuzzy control systems," *Fuzzy Sets Syst.*, vol. 14, pp. 199–210, 1984.

[115] ____, "An annotated bibliography of fuzzy control," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 249–269.

[116] E. Trillas and L. Valverde, "On mode and implication in approximate reasoning," in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Eds. Amsterdam: North-Holland, 1985, pp. 157–166.

[117] Y. Tsukamoto, "An approach to fuzzy reasoning method," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragade, and R. R. Yager, Eds. Amsterdam: North-Holland, 1979.

[118] I. G. Umbers and P. J. King, "An analysis of human-decision making in cement kiln control and the implications for automation," *Int. J. Man Mach. Studies*, vol. 12, no. 1, pp. 11–23, 1980.

[119] M. Uragami, M. Mizumoto, and K. Tananka, "Fuzzy robot controls," *Cybern.*, vol. 6, pp. 39–64, 1976.

[120] H. Watanabe and W. Dettloff, "Reconfigurable fuzzy logic processor: A full custom digital VLSI," in *Int. Workshop on Fuzzy System Applications*, Iizuka, Japan, Aug. 1988, pp. 49–50.

[121] D. Willaeys, "Optimal control of fuzzy systems," in *Proc. Int. Congress on Applied Systems Research and Cybernetics*, Acapulco, Dec. 1980.

[122] ____, "Decision making with possibilistic dynamic systems," in *Proc. 20th IEEE Conf. on Decision and Control*, San Diego, Dec. 1981.

[123] D. Willaeys and N. Malvache, "Use of fuzzy model for process control," in *Proc. IEEE Int. Conf. on Cybernetics and Society*, 1978, pp. 942–946.

[124] ____, "Contribution of the fuzzy sets theory to man-machine system," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragade, and R. R. Yager, Eds. Amsterdam: North-Holland, 1979, pp. 481–500.

[125] C. W. Xu and Y. Zailu, "Fuzzy model identification and self-learning for dynamic systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-17, no. 4, pp. 683–689, 1987.

[126] R. R. Yager, S. Ovchinnikov, R. M. Tong, and H. T. Nguyen, *Fuzzy Sets and Applications: Selected Papers by L. A. Zadeh.* New York: Wiley-Interscience, 1987.

[127] O. Yagishita, O. Itoh, and M. Sugeno, "Application of fuzzy reasoning to the water purification process," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 19–40.

[128] T. Yamakawa and T. Miki, "The current mode fuzzy logic integrated circuits fabricated by the standard CMOS process," *IEEE Trans. Computer*, vol. C-35, no. 2, pp. 161–167, 1986.

[129] T. Yamakawa and K. Sasaki, "Fuzzy memory device," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 551–555.

[130] T. Yamakawa, "High speed fuzzy controller hardware system," in *Proc. 2nd Fuzzy System Symp.*, Japan, 1986, pp. 122–130.

[131] ____, "Fuzzy controller hardware system," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987.

[132] ____, "A simple fuzzy computer hardware system employing min and max operations—A challenge to 6th generation computer," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987.

[133] T. Yamakawa, "Fuzzy microprocessors—Rule chip and defuzzifier chip," in *Int. Workshop on Fuzzy System Applications*, Iizuka, Japan, Aug. 1988, pp. 51–52.

[134] T. Yamazaki and M. Sugeno, "A microprocessor based fuzzy controller for industrial purposes," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 231–240.

[135] S. Yasunobu, S. Miyamoto, and H. Ihara, "Fuzzy control for automatic train operation system," in *Proc. 4th IFAC/IFIP/IFORS Int. Congress on Control in Transportation Systems*, Baden-Baden, April 1983.

[136] S. Yasunobu and S. Miyamoto, "Automatic train operation by predictive fuzzy control," in *Industrial Application of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 1–18.

[137] S. Yasunobu and T. Hasegawa, "Evaluation of an automatic container crane operation system based on predictive fuzzy control," *Control Theory Adv. Technol.*, vol. 2, no. 3, pp. 419–432, 1986.

[138] ____, "Predictive fuzzy control and its application for automatic container crane operation system," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 349–352.

[139] S. Yasunobu, S. Sekino, and T. Hasegawa, "Automatic train operation and automatic crane operation systems based on predictive fuzzy control," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 835–838.

[140] M. Yonekara, "The application of fuzzy sets theory to the temperature of box annealing furnaces using simulation techniques," in *Proc. 8th IFAC World Congress*, Tokyo, 1981.

[141] L. A. Zadeh, "Fuzzy sets," *Informat. Control*, vol. 8, pp. 338–353, 1965.

[142] ____, "Fuzzy algorithm," *Informat. Control*, vol. 12, pp. 94–102, 1968.

[143] ____, "Toward a theory of fuzzy systems," in *Aspects of Network*

and System Theory, R. E. Kalman and N. DeClaris, Ed. New York: Holt, Rinehart and Winston, 1971, pp. 469–490.

[144] ____, "Similarity relations and fuzzy orderings," *Informat. Sci.*, vol. 3, pp. 177–200, 1971.

[145] ____, "A rationale for fuzzy control," *Trans. ASME, J. Dynam. Syst. Measur. Control*, vol. 94, pp. 3–4, 1972.

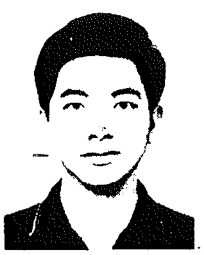
[146] ____, "Outline of a new approach to the analysis complex systems and decision processes," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-3, pp. 28–44, 1973.

[147] ____, "The concept of a linguistic variable and its application to approximate reasoning I, II, III," *Informat. Sci.*, vol. 8, pp. 199–251, pp. 301–357; vol. 9, pp. 43–80, 1975.

[148] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*. Boston: Kluwer Academic Publishers, 1985.

[149] H. J. Zimmermann, *Fuzzy Sets, Decision Making, and Expert Systems*. Boston: Kluwer Academic Publishers, 1987.

[150] C. C. Lee, "Fuzzy logic in control systems: Fuzzy logic controller —Part II," *IEEE Trans. Syst. Man Cybern.*, vol. 20, no. 2, pp. 419–435, Mar./Apr. 1990.



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Fuzzy Logic in Control Systems: Fuzzy Logic Controller, Part II

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Abstract —**During the past several years, fuzzy control has emerged as one of the most active and fruitful areas for research in the applications of fuzzy set theory, especially in the realm of industrial processes, which do not lend themselves to control by conventional methods because of a lack of quantitative data regarding the input–output relations. Fuzzy control is based on fuzzy logic—a logical system that is much closer in spirit to human thinking and natural language than traditional logical systems. The fuzzy logic controller (FLC) based on fuzzy logic provides a means of converting a linguistic control strategy based on expert knowledge into an automatic control strategy. A survey of the FLC is presented; a general methodology for constructing an FLC and assessing its performance is described; and problems that need further research are pointed out. In particular, the exposition includes a discussion of fuzzification and defuzzification strategies, the derivation of the database and fuzzy control rules, the definition of fuzzy implication, and an analysis of fuzzy reasoning mechanisms.**

I. DECISIONMAKING LOGIC

AS WAS noted in Part I of this paper [150], an FLC may be regarded as a means of emulating a skilled human operator. More generally, the use of an FLC may be viewed as still another step in the direction of modeling human decisionmaking within the conceptual framework of fuzzy logic and approximate reasoning. In this context, the forward data-driven inference (generalized modus ponens) plays an especially important role. In what follows, we shall investigate fuzzy implication functions, the sentence connectives *and* and *also*, compositional operators, inference mechanisms, and other concepts that are closely related to the decisionmaking logic of an FLC.

A. Fuzzy Implication Functions

In general, a fuzzy control rule is a fuzzy relation which is expressed as a fuzzy implication. In fuzzy logic, there are many ways in which a fuzzy implication may be defined. The definition of a fuzzy implication may be expressed as a fuzzy implication function. The choice of a fuzzy implication function reflects not only the intuitive criteria for implication but also the effect of connective *also*.

1) Basic Properties of a Fuzzy Implication Function: The choice of a fuzzy implication function involves a number of criteria, which are discussed in [3], [24], [2], [71], [18], [52], [19], [116], [85], [72], and [96]. In particular, Baldwin and Pilsworth [3] considered the following basic characteristics of a fuzzy implication function: fundamental property, smoothness property, unrestricted inference, symmetry of generalized modus ponens and generalized modus tollens, and a measure of propagation of fuzziness. All of these properties are justified on purely intuitive grounds. We prefer to say that the inference (consequence) should be as close as possible to the input truth function value, rather than be equal to it. This gives us a more flexible criterion for choosing a fuzzy implication function. Furthermore, in a chain of implications, it is necessary to consider the "fuzzy syllogism" [147] associated with each fuzzy implication function before we can talk about the propagation of fuzziness.

Fukami, Mizumoto, and Tanaka [24] have proposed a set of intuitive criteria for choosing a fuzzy implication function that constrains the relations between the antecedents and consequents of a conditional proposition, with the latter playing the role of a premise in approximate reasoning. As is well known, there are two important fuzzy implication inference rules in approximate reasoning. They are the generalized modus ponens (GMP) and the generalized modus tollens (GMT). Specifically,

$$\begin{array}{ll} \text{premise 1: } x \text{ is } A' & \\ \underline{\text{premise 2: if } x \text{ is } A \text{ then } y \text{ is } B} & \text{(GMP)} \\ \text{consequence: } y \text{ is } B' & \end{array}$$

$$\begin{array}{ll} \text{premise 1: } y \text{ is } B' & \\ \underline{\text{premise 2: if } x \text{ is } A \text{ then } y \text{ is } B} & \text{(GMT)} \\ \text{consequence: } x \text{ is } A' & \end{array}$$

in which A, A', B, and B' are fuzzy predicates. The propositions above the line are the premises; and the proposition below the line is the consequence. The proposed criteria are summarized in Tables I and II. We note that if a causal relation between "x is A" and "y is B" is not strong in a fuzzy implication, the satisfaction of criterion 2-2 and criterion 3-2 is allowed. Criterion 4-2 is interpreted as: if x is A then y is B, else y is not B.

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TABLE I
INTUITIVE CRITERIA RELATING PRE1 AND CONS FOR GIVEN PRE2 IN GMP

	x is A'(Pre1)	y is B'(Cons)
Criterion 1	x is A	y is B
Criterion 2-1	x is very A	y is very B
Criterion 2-2	x is very A	y is B
Criterion 3-1	x is more or less A	y is more or less B
Criterion 3-2	x is more or less A	y is B
Criterion 4-1	x is not A	y is unknown
Criterion 4-2	x is not A	y is not B

TABLE II
INTUITIVE CRITERIA RELATING PRE1 AND CONS FOR GIVEN PRE2 IN GMT

	y is B'(Pre1)	x is A'(Cons)
Criterion 5	y is not B	x is not A
Criterion 6	y is not very B	x is not very A
Criterion 7	y is not more or less B	x is not more or less A
Criterion 8-1	y is B	x is unknown
Criterion 8-2	y is B	x is A

Although this relation is not valid in formal logic, we often make such an interpretation in everyday reasoning. The same applies to criterion 8.

2) Families of Fuzzy Implication Functions: Following Zadeh's [146] introduction of the compositional rule of inference in approximate reasoning, a number of researchers have proposed various implication functions in which the antecedents and consequents contain fuzzy variables. Indeed, nearly 40 distinct fuzzy implication functions have been described in the literature. In general, they can be classified into three main categories: the *fuzzy conjunction*, the *fuzzy disjunction*, and the *fuzzy implication*. The former two bear a close relation to a fuzzy Cartesian product. The latter is a generalization of implication in multiple-valued logic and relates to the extension of material implication, implication in propositional calculus, modus ponens, and modus tollens [18]. In what follows, after a short review of triangular norms and triangular co-norms, we shall give the definitions of fuzzy conjunction, fuzzy disjunction, and fuzzy implication. Some fuzzy implication functions, which are often employed in an FLC and are commonly found in the literature, will be derived.

Definition 1: Triangular Norms: The triangular norm $*$ is a two-place function from $[0,1]\times[0,1]$ to $[0,1]$, i.e., $*$: $[0,1]\times[0,1]\rightarrow[0,1]$, which includes intersection, algebraic product, bounded product, and drastic product. The greatest triangular norm is the intersection and the least one is the drastic product. The operations associated with triangular norms are defined for all $x,y\in[0,1]$:

intersection	$x \wedge y = \min\{x,y\}$
algebraic product	$x \cdot y = xy$
bounded product	$x \odot y = \max\{0, x+y-1\}$
drastic product	$x \cap y = \begin{cases} x & y=1 \\ y & x=1 \\ 0 & x,y<1. \end{cases}$

Definition 2: Triangular Co-Norms: The triangular co-norms $\dotplus$ is a two-place function from $[0,1]\times[0,1]$ to $[0,1]$, i.e. $\dotplus$: $[0,1]\times[0,1]$ to $[0,1]$, which includes union, algebraic sum, bounded sum, drastic sum, and disjoint sum. The operations associated with triangular co-norms are defined for all $x,y\in[0,1]$:

union	$x \vee y = \max\{x,y\}$
algebraic sum	$x \dotplus y = x+y-xy$
bounded sum	$x \oplus y = \min\{1, x+y\}$
drastic sum	$x \cup y = \begin{cases} x & y=0 \\ y & x=0 \\ 1 & x,y>0 \end{cases}$
disjoint sum	$x \Delta y = \max\{\min(x, 1-y), \min(1-x, y)\}.$

The triangular norms are employed for defining conjunctions in approximate reasoning, while the triangular co-norms serve the same role for disjunctions. A fuzzy control rule, "if x is A then y is B," is represented by a fuzzy implication function and is denoted by $A\rightarrow B$, where A and B are fuzzy sets in universes U and V with membership functions μ_A and μ_B, respectively.

Definition 3: Fuzzy Conjunction: The fuzzy conjunction is defined for all $u\in U$ and $v\in V$ by

$$A\rightarrow B = A\times B = \int_{U\times V} \mu_A(u) * \mu_B(v)/(u,v)$$

where $*$ is an operator representing a triangular norm.

Definition 4: Fuzzy Disjunction: The fuzzy disjunction is defined for all $u\in U$ and $v\in V$ by

$$A\rightarrow B = A\times B = \int_{U\times V} \mu_A(u) \dotplus \mu_B(v)/(u,v)$$

where $\dotplus$ is an operator representing a triangular co-norm.

Definition 5: Fuzzy Implication: The fuzzy implication is associated with five families of fuzzy implication functions in use. As before, $*$ denotes a triangular norm and $\dotplus$ is a triangular co-norm.

4.1) Material implication:

$$A\rightarrow B = (\text{not } A) \dotplus B$$

4.2) Propositional calculus:

$$A\rightarrow B = (\text{not } A) \dotplus (A * B)$$

4.3) Extended propositional calculus:

$$A\rightarrow B = (\text{not } A \times \text{not } B) \dotplus B$$

4.4) Generalization of modus ponens:

$$A\rightarrow B = \sup\{c\in[0,1],\ A * c \leqslant B\}$$

4.5) Generalization of modus tollens:

$$A \rightarrow B = \inf\{t \in [0,1],\ B \dot{+} t \leqslant A\}$$

Based on these definitions, many fuzzy implication functions may be generated by employing the triangular norms and co-norms. For example, by using the definition of the fuzzy conjunction, Mamdani's mini-fuzzy implication, R_c, is obtained if the intersection operator is used. Larsen's product fuzzy implication, R_p, is obtained if the algebraic product is used. Furthermore, R_{bp} and R_{dp} are obtained if the bounded product and the drastic product are used, respectively. The following fuzzy implications, which are often adopted in an FLC, will be discussed in more detail at a later point.

Mini-operation rule of fuzzy implication [Mamdani]:

$$\begin{aligned} R_c &= A \times B \\ &= \int_{U \times V} \mu_A(u) \wedge \mu_B(v)/(u,v). \end{aligned}$$

Product operation rule of fuzzy implication [Larsen]:

$$\begin{aligned} R_p &= A \times B \\ &= \int_{U \times V} \mu_A(u)\mu_B(v)/(u,v). \end{aligned}$$

Arithmetic rule of fuzzy implication [Zadeh]:

$$\begin{aligned} R_a &= (\text{not } A \times V) \oplus (U \times B) \\ &= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u,v). \end{aligned}$$

Maxmin rule of fuzzy implication [Zadeh]:

$$\begin{aligned} R_m &= (A \times B) \cup (\text{not } A \times V) \\ &= \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u,v). \end{aligned}$$

Standard sequence fuzzy implication:

$$\begin{aligned} R_s &= A \times V \rightarrow U \times B \\ &= \int_{U \times V} (\mu_A(u) > \mu_B(v))/(u,v) \end{aligned}$$

where

$$\mu_A(u) > \mu_B(v) = \begin{cases} 1 & \mu_A(u) \leqslant \mu_B(v) \\ 0 & \mu_A(u) > \mu_B(v). \end{cases}$$

Boolean fuzzy implication:

$$\begin{aligned} R_b &= (\text{not } A \times V) \cup (U \times B) \\ &= \int_{U \times V} (1 - \mu_A(u)) \vee (\mu_B(v))/(u,v). \end{aligned}$$

Goguen's fuzzy implication:

$$\begin{aligned} R_\Delta &= A \times V \rightarrow U \times B \\ &= \int_{U \times V} (\mu_A(u) \gg \mu_B(v))/(u,v) \end{aligned}$$

where

$$\mu_A(u) \gg \mu_B(v) = \begin{cases} 1 & \mu_A(u) \leqslant \mu_B(v) \\ \dfrac{\mu_B(u)}{\mu_A(v)} & \mu_A(u) > \mu_B(v). \end{cases}$$

We note that Zadeh's arithmetic rule follows from Definition 5.1 by using the bounded sum operator; Zadeh's maxmin rule follows from Definition 5.2 by using the intersection and union operators; the standard sequence implication follows from Definition 5.4 by using the bounded product; Boolean fuzzy implication follows from Definition 5.1 by using the union; and Goguen's fuzzy implication follows from Definition 5.4 by using the algebraic product.

3) Choice of a Fuzzy Implication Function: First, we investigate the consequences resulting from applying the preceding forms of fuzzy implication in fuzzy inference and, in particular, the GMP and GMT. The inference is based on the sup–min compositional rule of inference. In the GMP, we examine the consequence of the following compositional equation:

$$B' = A' \circ R$$

where

R fuzzy implication (relation),
$\circ$ sup–min compositional operator,
A' a fuzzy set which has the form:
$A = \int_U \mu_A(u)/u$
very $A = A^2 = \int_U \mu_A^2(u)/u$
more or less $A = A^{0.5} = \int_U \mu_A^{0.5}(u)/u$
not $A = \int_U 1 - \mu_A(u)/u$.

Similarly, in the GMT, we examine the consequence of the following equation:

$$A' = R \circ B'$$

where

R fuzzy implication (relation)
B' a fuzzy set that has the form:
not $B = \int_V 1 - \mu_B(u)/u$
not very $B = \int_V 1 - \mu_B^2(v)/v$
not more or less $B = \int_V 1 - \mu_B^{0.5}(v)/v$
$B = \int_V \mu_B(v)/v$.

The Case of R_p: Larsen's Product Rule: A method for computing the generalized modus ponens and the generalized modus tollens laws of inference is described in [3]. The graphs corresponding to Larsen's fuzzy implication R_p are given in Fig. 1. The graph with parameter μ_A is used for the GMP, and the graph with μ_B is used for the GMT.

Larsen's Product Rule in GMP: Suppose that $A' = A^\alpha$ ($\alpha > 0$); then the consequence B_p' is inferred as follows:

$$\begin{aligned} B_p' &= A^\alpha \circ R_p \\ &= \int_U \mu_A^\alpha(u)/u \circ \int_{U \times V} \mu_A(u) \cdot \mu_B(v)/(u,v). \end{aligned}$$

The membership function $\mu_{B_p'}$ of the fuzzy set B_p' is pointwise defined for all $v \in V$ by

$$\begin{aligned} \mu_{B_p'}(v) &= \sup_{u \in U} \min\{\mu_A^\alpha(u), \mu_A(u)\mu_B(v)\} \\ &= \sup_{u \in U} S_p(1 - \mu_A^\alpha(u)) \end{aligned}$$

where

$$S_p(\mu_A^\alpha(u)) \triangleq \min\{\mu_A^\alpha(u), \mu_A(u)\mu_B(v)\}.$$

$\{A' = A\}$: The values of $S_p(\mu_A(u))$ with a parameter $\mu_B(v)$, say $\mu_B(v) = 0.3$ and 0.8, are indicated in Fig. 2 by a broken line and dotted line, respectively. The member-

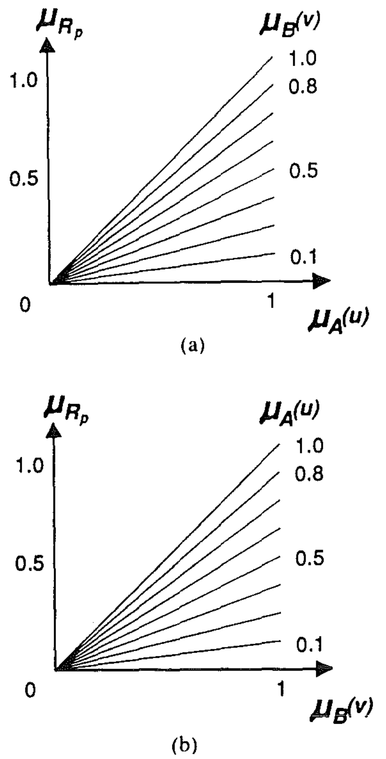

Fig. 1. Diagrams for calculation of membership functions. (a) μ_{R_p} versus μ_A with the parameter μ_B. (b) μ_{R_p} versus μ_B with parameter μ_A.

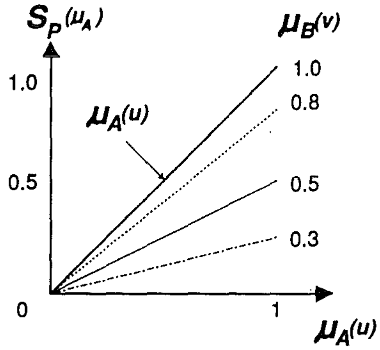

Fig. 2. Approximate reasoning: generalized modus ponens with Larsen's product operation rule.

ship function $\mu_{B'_p}$ is obtained by

$$\begin{aligned}\mu_{B'_p}(v) &= \sup_{u \in U} \min\{\mu_A(u), \mu_A(u)\mu_B(v)\}\\ &= \sup_{u \in U} \mu_A(u)\mu_B(v)\\ &= \mu_B(v), \qquad \mu_A(u) = 1.\end{aligned}$$

{A' = A²}: The values of $S_p(\mu_A^2(u))$ with a parameter $\mu_B(v)$, say $\mu_B(v) = 0.3$ and 0.8, are indicated in Fig. 3 by a broken line and dotted line, respectively. The membership function $\mu_{B'_p}$ may be expressed as

$$\begin{aligned}\mu_{B'_p}(v) &= \sup_{u \in U} \min\{\mu_A^2(u), \mu_A(u)\mu_B(v)\}\\ &= \mu_B(v).\end{aligned}$$

{A' = A^0.5}: The values of $S_p(\mu_A^{0.5}(u))$ with a parameter $\mu_B(v)$, say $\mu_B(v) = 0.3$ and 0.8, are indicated in Fig. 4 by a broken line and dotted line, respectively. The membership function $\mu_{B'_p}$ is given by

$$\begin{aligned}\mu_{B'_p}(v) &= \sup_{u \in U} \min\{\mu_A^{0.5}(u), \mu_A(u)\mu_B(v)\}\\ &= \mu_B(v).\end{aligned}$$

{A' = not A}: The values of $S_p(-\mu_A(u))$ with a parameter $\mu_B(v)$, say $\mu_B(v) = 0.3$ and 0.8, are indicated in Fig. 5 by a broken line and dotted line, respectively. The membership function $\mu_{B'_p}$ is given by

$$\begin{aligned}\mu_{B'_p}(v) &= \sup_{u \in U} \min\{1 - \mu_A(u), \mu_A(u)\mu_B(v)\}\\ &= \frac{\mu_B(v)}{1 + \mu_B(v)}.\end{aligned}$$

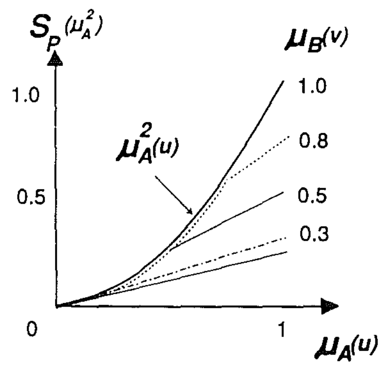

Fig. 3. Approximate reasoning: generalized modus ponens with Larsen's product operation rule.

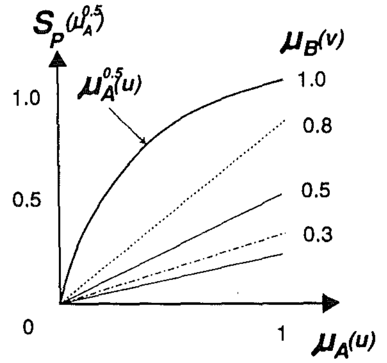

Fig. 4. Approximate reasoning: generalized modus ponens with Larsen's product operation rule.

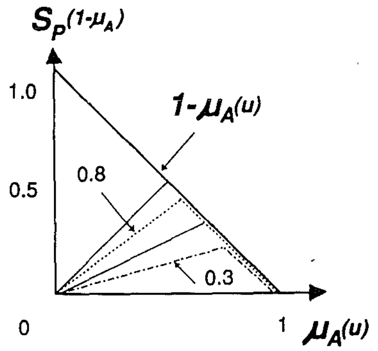

Fig. 5. Approximate reasoning: generalized modus ponens with Larsen's product operation rule.

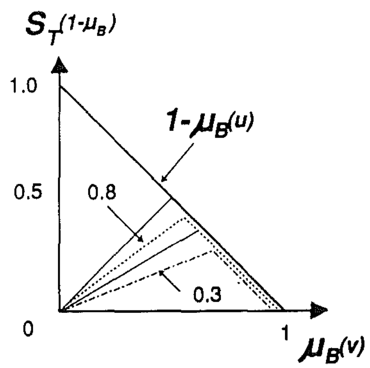

Fig. 6. Approximate reasoning: generalized modus tollens with Larsen's product operation rule.

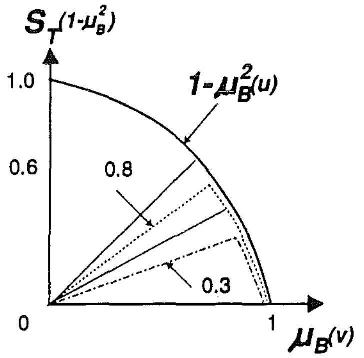

Fig. 7. Approximate reasoning: generalized modus tollens with Larsen's product operation rule.

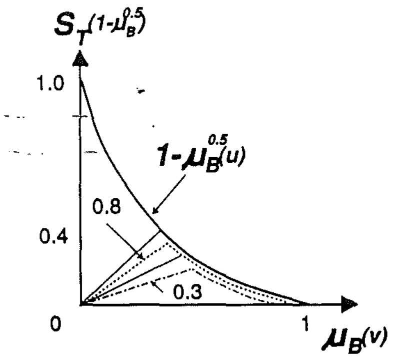

Fig. 8. Approximate reasoning: generalized modus tollens withLarsen's product operation rule.

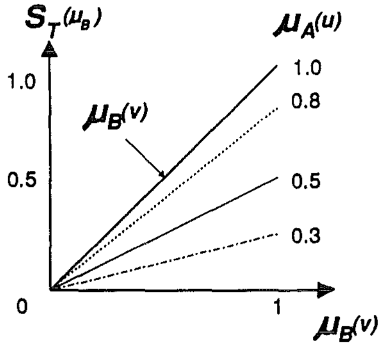

Fig. 9. Approximate reasoning: generalized modus tollens with Larsen's product operation rule.

TABLE III
SUMMARY OF INFERENCE RESULTS FOR GENERALIZED MODUS PONENS

	A	Very A	More or Less A	Not A
R_c	μ_B	μ_B	μ_B	$0.5\wedge\mu_B$
R_p	μ_B	μ_B	μ_B	$\dfrac{\mu_B}{1+\mu_B}$
R_a	$\dfrac{1+\mu_B}{2}$	$\dfrac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}$	$\dfrac{\sqrt{5+4\mu_B}-1}{2}$	1
R_m	$0.5\vee\mu_B$	$\dfrac{3-\sqrt{5}}{2}\vee\mu_B$	$\dfrac{\sqrt{5}-1}{2}\vee\mu_B$	1
R_b	$0.5\vee\mu_B$	$\dfrac{3-\sqrt{5}}{2}\vee\mu_B$	$\dfrac{\sqrt{5}-1}{2}\vee\mu_B$	1
R_s	μ_B	μ_B^2	$\sqrt{\mu_B}$	1
R_Δ	$\sqrt{\mu_B}$	$\mu_B^{2/3}$	$\mu_B^{1/3}$	1

Larsen's Product Rule in GMT: Suppose that $B' =$ not B^α $(\alpha > 0)$; then the consequence A'_p is inferred as follows:

$$A'_t = R_p \circ (\text{not } B^\alpha)$$

$$= \int_{U\times V} \mu_A(u)\mu_B(v) \Big/ (u,v) \circ \int_V (1-\mu_B^\alpha(v))/v.$$

The membership function $\mu_{A'_t}$ of the fuzzy set A'_t is pointwise defined for all $u \in U$ by

$$\mu_{A'_t}(u) = \sup_{v\in V} \min\{\mu_A(u)\mu_B(v), 1-\mu_B^\alpha(v)\}$$

$$= \sup_{v\in V} S_t(1-\mu_B^\alpha(v))$$

where

$$S_t(1-\mu_B^\alpha(v)) \triangleq \min\{\mu_A(u)\mu_B(v), 1-\mu_B^\alpha(v)\}.$$

{B' = not B}: The values of $S_t(1-\mu_B(v))$ with a parameter $\mu_A(u)$, say $\mu_A(u) = 0.3$ and 0.8, are indicated in Fig. 6 by a broken line and dotted line, respectively. The membership function $\mu_{A'_t}$ is given by

$$\mu_{A'_t}(u) = \sup_{v\in V} \min\{\mu_A(u)\mu_B(v), 1-\mu_B(v)\}$$

$$= \frac{\mu_A(u)}{1+\mu_A(u)}.$$

{B' = not B²}: The values of $S_t(1-\mu_B^2(v))$ with a parameter $\mu_A(u)$, say $\mu_A(u) = 0.3$ and 0.8, are indicated in Fig. 7 by a broken line and dotted line, respectively. The membership function $\mu_{A'_t}$ is given by

$$\mu_{A'_t}(u) = \sup_{v\in V} \min\{\mu_A(u)\mu_B(v), 1-\mu_B^2(v)\}$$

$$= \frac{\mu_A(u)\sqrt{\mu_A^2(u)+4}-\mu_A(u)}{2}.$$

{B' = not B⁰·⁵}: The values of $S_t(1-\mu_B^{0.5}(v))$ with a parameter $\mu_A(u)$, say $\mu_A(u) = 0.3$ and 0.8, are indicated in Fig. 8 by a broken line and dotted line, respectively. The

TABLE IV
SUMMARY OF INFERENCE RESULTS FOR GENERALIZED MODUS TOLLENS

	Not B	Not Very B	Not More or Less B	B
R_c	$0.5 \wedge \mu_A$	$\frac{\sqrt{5}-1}{2} \wedge \mu_A$	$\frac{3-\sqrt{5}}{2} \wedge \mu_A$	μ_A
R_p	$\frac{\mu_A}{1+\mu_A}$	$\frac{\mu_A\sqrt{\mu_A^2+4}-\mu_A}{2}$	$\frac{2\mu_A+1-\sqrt{4\mu_A+1}}{2\mu_A}$	μ_A
R_a	$1-\frac{\mu_A}{2}$	$\frac{1-2\mu_A+\sqrt{1+4\mu_A}}{2}$	$\frac{3-\sqrt{1+\mu_A}}{2}$	1
R_m	$0.5 \vee (1-\mu_A)$	$(1-\mu_A) \vee \left(\frac{\sqrt{5}-1}{2} \wedge \mu_A\right)$	$\frac{3-\sqrt{5}}{2} \vee (1-\mu_A)$	$\mu_A \vee (1-\mu_A)$
R_b	$0.5 \vee (1-\mu_A)$	$\frac{\sqrt{5}-1}{2} \vee (1-\mu_A)$	$\frac{3-\sqrt{5}}{2} \vee (1-\mu_A)$	1
R_s	$1-\mu_A$	$1-\mu_A^2$	$1-\sqrt{\mu_A}$	1
μ_Δ	$\frac{1}{1+\mu_A}$	$\frac{\sqrt{1+4\mu_A^2}-1}{2\mu_A^2}$	$\frac{2+\mu_A-\sqrt{\mu_A^2+4\mu_A}}{2}$	1

TABLE V
SATISFACTION OF VARIOUS FUZZY IMPLICATION FUNCTIONS UNDER INTUITIVE CRITERIA

	R_c	R_p	R_a	R_m	R_s	R_Δ	R_b
Criteria 1	○	○	×	×	○	×	×
Criteria 2-1	×	×	×	×	○	×	×
Criteria 2-2	○	○	×	×	×	×	×
Criteria 3-1	×	×	×	×	○	×	×
Criteria 3-2	○	○	×	×	×	×	×
Criteria 4-1	×	×	○	○	○	○	○
Criteria 4-2	×	×	×	×	×	×	×
Criteria 5	×	×	×	×	○	×	×
Criteria 6	×	×	×	×	○	×	×
Criteria 7	×	×	×	×	○	×	×
Criteria 8-1	×	×	○	×	○	○	○
Criteria 8-2	○	○	×	×	×	×	×

membership function $\mu_{A_t'}$ is given by

$$\mu_{A_t'}(u) = \sup_{v \in V} \min\{\mu_A(u)\mu_B(v), 1-\mu_B^{0.5}(v)\}$$

$$= \frac{2\mu_A(u)+1-\sqrt{4\mu_A+4}}{2\mu_A(u)}.$$

{B' = B}: The values of $S_t(\mu_B(v))$ with a parameter $\mu_A(u)$, say $\mu_A(u) = 0.3$ and 0.8, are indicated in Fig. 9 by a broken line and dotted line, respectively. The membership function $\mu_{A_t'}$ is given by

$$\mu_{A_t'}(u) = \sup_{v \in V} \{\mu_A(u)\mu_B(v), \mu_B(v)\}$$

$$= \mu_A(u).$$

The remaining consequences [24] inferred by $R_a, R_c, R_m, R_s, R_b, R_\Delta$ can be obtained by the same method as just described. The results are summarized in Tables III and IV.

By employing the intuitive criteria in Tables I and II in Tables III and IV, we can determine how well a fuzzy implication function satisfies them. This information is summarized in Table V.

In FLC applications, a control action is determined by the observed inputs and the control rules, without the consequent of one rule serving as the antecedent of another. In effect, the FLC functions as a one-level forward data-driven inference (GMP). Thus the backward goal-driven inference (GMT), chaining inference mechanisms (syllogisms), and contraposition do not play a role in the FLC, since there is no need to infer a fuzzy control action through the use of these inference mechanisms.

Although R_c and R_p do not have a well-defined logical structure, the results tabulated in Table V indicate that they are well suited for approximate reasoning, especially for the generalized modus ponens.

R_m has a logical structure which is similar to R_b. R_a is based on the implication rule in Lukasiewicz's logic L_{Aleph}. However, R_m and R_a are not well suited for approximate reasoning since the inferred consequences do not always fit our intuition. Furthermore, for multiple-valued logical systems, R_b and R_Δ have significant shortcomings. Overall, R_s yields reasonable results and thus constitutes an appropriate choice for use in approximate reasoning.

B. Interpretation of Sentence Connectives "and, also"

In most of the existing FLC's, the sentence connective "and" is usually implemented as a fuzzy conjunction in a Cartesian product space in which the underlying variables take values in different universes of discourse. As an illustration, in "if (A and B) then C," the antecedent is interpreted as a fuzzy set in the product space $U \times V$, with the membership function given by

$$\mu_{A \times B}(u,v) = \min\{\mu_A(u), \mu_B(v)\}$$

or

$$\mu_{A \times B}(u,v) = \mu_A(u) \cdot \mu_B(v)$$

where U and V are the universes of discourse associated with A and B, respectively.

When a fuzzy system is characterized by a set of fuzzy control rules, the ordering of the rules is immaterial. This necessitates that the sentence connective "also" should have the properties of commutativity and associativity (see sections III-A and III-C in Part I and Part D in this section). In this connection, it should be noted that the operators in triangular norms and co-norms possess these properties and thus qualify as the candidates for the interpretation of the connective "also." In general, we use the triangular co-norms in association with fuzzy conjunction and disjunction, and the triangular norms in association with fuzzy implication. The experimental results [52]–[54], [96], [73] and the theoretical studies [18], [85], [116], [19] relate to this issue.

Kiszka *et al.* [52] described a preliminary investigation of the fuzzy implication functions and the sentence connective "also" in the context of the fuzzy model of a dc series motor. In later work, they presented additional results for fuzzy implication functions and the connective "also" in terms of the union and intersection operators [53], [54].

Our investigation leads to some preliminary conclusions. First, the connective "also" has a substantial influence on the quality of a fuzzy model, as we might expect. Fuzzy implication functions such as R_s, R_Δ, and R_a with the connective "also" defined as the union operator, and R_c, R_p, R_{bp}, and R_{dp} defined as the intersection, yield satisfactory results. These fuzzy implication functions differ in the number of mathematical operations which are needed for computer implementation.

Recently, Stachowicz and Kochanska [96] studied the characteristics of 38 types of fuzzy implication along with nine different interpretations (in terms of triangular norms and co-norms) of the connective "also," based on various forms of the operational curve of a series motor. Based on their results, we tabulate in Table VI a summary of the most appropriate pairs for the FLC of the fuzzy implication function and the connective "also."

Additional results relating to the interpretation of the connective "also" as the union and the intersection are reported in [73]. The investigation in question is based on a plant model with first-order delay. It is established that the fuzzy implication functions R_c, R_p, R_{bp}, R_{dp} with the connective "also" as the union operator yield the best control results. Furthermore, the fuzzy implications R_s and R_g are not well suited for control applications even though they yield reasonably good results in approximate reasoning.

TABLE VI

SUITABLE PAIRS OF A FUZZY IMPLICATION FUNCTION AND CONNECTIVE "*also*"

Implication Rule	Connective Also
$R_c R_p R_{bp} R_{dp}$	$\cup\ \hat{+}\ \oplus\ \dot{\cup}\ \Delta$
R_a	$\cap \cdot \odot\ \dot{\cap}$
R_m	–
$R_s R_\Delta R_g$	$(\cap \cdot \odot\ \dot{\cap}^a$
R_b	$\cdot \odot\ \dot{\cap}$

^{a}It depends on the shape of reproduced curve which forms the set of fuzzy control rules.

From a practical point of view, the computational aspects of an FLC require a simplification of the fuzzy control algorithm. In this perspective, Mamdani's R_c and Larsen's R_p with the connective "also" as the union operator appear to be better suited for constructing fuzzy models than the other methods in FLC applications. We will have more to say about these methods at a later point.

C. Compositional Operators

In a general form, a compositional operator may be expressed as the sup–star composition, where "star" denotes an operator—e.g., min, product, etc.—which is chosen to fit a specific application. In the literature, four kinds of compositional operators can be used in the compositional rule of inference, namely:

sup–min operation [Zadeh, 1973],
sup–product operation [Kaufmann, 1975],
sup–bounded-product operation [Mizumoto, 1981],
sup–drastic-product operation [Mizumoto, 1981].

In FLC applications, the sup–min and sup–product compositional operators are the most frequently used. The reason is obvious, when the computational aspects of an FLC are considered. However, interesting results can be obtained if we apply the sup–product, sup–bounded-product, and sup–drastic-product operations with different fuzzy implication functions in approximate reasoning [70], [72]. The inferred results employing these compositional operators are better than those employing the sup–min operator. Further investigation of these issues in the context of the accuracy of fuzzy models may provide interesting results.

D. Inference Mechanisms

The inference mechanisms employed in an FLC are generally much simpler than those used in a typical expert system, since in an FLC the consequent of a rule is not applied to the antecedent of another. In other words, in FLC we do not employ the chaining inference mechanism, since the control actions are based on one-level forward data-driven inference (GMP).

The rule base of an FLC is usually derived from expert knowledge. Typically, the rule base has the form of a

MIMO system

$$R = \{R^1_{\text{MIMO}}, R^2_{\text{MIMO}}, \cdots, R^n_{\text{MIMO}}\}$$

where R^i_{MIMO} represents the rule: if (x is A_i and $\cdots$, and y is B_i) then (z_1 is $C_i, \cdots, z_q$ is D_i). The antecedent of R^i_{MIMO} forms a fuzzy set $A_i \times \cdots \times B_i$ in the product space $U \times \cdots \times V$. The consequent is the union of q independent control actions. Thus the ith rule R^i_{MIMO} may be represented as a fuzzy implication

$$R^i_{\text{MIMO}}: (A_i \times \cdots \times B_i) \to (z_1 + \cdots + z_q)$$

from which it follows that the rule base R may be represented as the union

$$\begin{aligned} R &= \left\{ \bigcup_{i=1}^{n} R^i_{\text{MIMO}} \right\} \\ &= \left\{ \bigcup_{i=1}^{n} \left[(A_i \times \cdots \times B_i) \to (z_1 + \cdots + z_q) \right] \right\} \\ &= \left\{ \bigcup_{i=1}^{n} \left[(A_i \times \cdots \times B_i) \to z_1) \right], \right. \\ &\quad \cdot \bigcup_{i=1}^{n} \left[(A_i \times \cdots \times B_i) \to z_2) \right], \cdots, \\ &\quad \left. \bigcup_{i=1}^{n} \left[(A_i \times \cdots \times B_i) \to z_q) \right] \right\} \\ &= \left\{ \bigcup_{k=1}^{q} \bigcup_{i=1}^{n} \left[(A_i \times \cdots \times B_i) \to z_k) \right] \right\} \\ &= \{RB^1_{\text{MISO}}, RB^2_{\text{MISO}}, \cdots, RB^q_{\text{MISO}}\}. \end{aligned}$$

In effect, the rule base R of an FLC is composed of a set of sub-rule-bases RB^i_{MISO}, with each sub-rule-base RB^i_{MISO} consisting of n fuzzy control rules with multiple process state variables and a single control variable. The general rule structure of a MIMO fuzzy system can therefore be represented as a collection of MISO fuzzy systems:

$$R = \{RB^1_{\text{MISO}}, RB^2_{\text{MISO}}, \cdots, RB^q_{\text{MISO}}\}$$

where RB^k_{MISO} represents the rule: if (x is A_i and $\cdots$, and y is B_i) then (z_k is D_i), $i = 1, 2, \cdots, n$.

Let us consider the following general form of MISO fuzzy control rules in the case of two-input/single-output fuzzy systems:

$$\begin{array}{rl} \text{input:} & x \text{ is } A' \text{ and } y \text{ is } B' \\ R_1: & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z \text{ is } C_1 \\ \text{also } R_2: & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z \text{ is } C_2 \\ & \cdots \\ & \cdots \\ \text{also } R_n: & \text{if } x \text{ is } A_n \text{ and } y \text{ is } B_n \text{ then } z \text{ is } C_n \\ \hline & z \text{ is } C' \end{array}$$

where x, y, and z are linguistic variables representing the process state variables and the control variable, respectively; A_i, B_i, and C_i are linguistic values of the linguistic variables x, y, and z in the universes of discourse U, V, and W, respectively, with $i = 1, 2, \cdots, n$.

The fuzzy control rule "if (x is A_i and y is B_i) then (z is C_i)" is implemented as a fuzzy implication (relation) R_i and is defined as

$$\begin{aligned} \mu_{R_i} &\triangleq \mu_{(A_i \text{ and } B_i \to C_i)}(u, v, w) \\ &= \left[\mu_{A_i}(u) \text{ and } \mu_{B_i}(v) \right] \to \mu_{C_i}(w) \end{aligned}$$

where "A_i and B_i" is a fuzzy set $A_i \times B_i$ in $U \times V$; $R_i \triangleq (A_i \text{ and } B_i) \to C_i$ is a fuzzy implication (relation) in $U \times V \times W$; and $\to$ denotes a fuzzy implication function.

The consequence C' is deduced from the sup-star compositional rule of inference employing the definitions of a fuzzy implication function and the connectives "and" and "also."

In what follows, we shall consider some useful properties of the FLC inference mechanism. First, we would like to show that the sup-min operator denoted by $\circ$ and the connective "also" as the union operator are commutative. Thus the fuzzy control action inferred from the complete set of fuzzy control rules is equivalent to the aggregated result derived from individual control rules. Furthermore, as will be shown later, the same properties are possessed by the sup-product operator. However, the conclusion in question does not apply when the fuzzy implication is used in its traditional logical sense [18], [19]. More specifically, we have

Lemma 1: $(A', B') \circ \bigcup_{i=1}^{n} R_i = \bigcup_{i=1}^{n} (A', B') \circ R_i$.

Proof:

$$\begin{aligned} C' &= (A', B') \circ \bigcup_{i=1}^{n} R_i \\ &= (A', B') \circ \bigcup_{i=1}^{n} (A_i \text{ and } B_i \to C_i). \end{aligned}$$

The membership function $\mu_{C'}$ of the fuzzy set C' is pointwise defined for all $w \in W$ by

$$\begin{aligned} \mu_{C'}(W) &= (\mu_{A'}(u), \mu_{B'}(v)) \circ \max_{u,v,w} \left(\mu_{R_1}(u,v,w), \right. \\ &\quad \left. \cdot \mu_{R_2}(u,v,w), \cdots, \mu_{R_n}(u,v,w) \right) \\ &= \sup_{u,v} \min \left\{ (\mu_{A'}(u), \mu_{B'}(v)), \max_{u,v,w} \left(\mu_{R_1}(u,v,w), \right. \right. \\ &\quad \left. \left. \cdot \mu_{R_2}(u,v,w), \cdots, \mu_{R_n}(u,v,w) \right) \right\} \\ &= \sup_{u,v} \max_{u,v,w} \left\{ \min \left[(\mu_{A'}(u), \mu_{B'}(v)), \mu_{R_1}(u,v,w) \right], \right. \\ &\quad \left. \cdots, \min \left[(\mu_{A'}(u), \mu_{B'}(v)), \mu_{R_n}(u,v,w) \right] \right\} \\ &= \max_{u,v,w} \left\{ \left[(\mu_{A'}(u), \mu_{B'}(v)) \circ \mu_{R_1}(u,v,w) \right], \right. \\ &\quad \left. \cdots, \left[(\mu_{A'}(u), \mu_{B'}(v)) \circ \mu_{R_n}(u,v,w) \right] \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} C' &= [(A',B')\circ R_1] \cup [(A',B')\circ R_2] \\ &\quad \cdot \cup \cdots \cup [(A',B')\circ R_n] \\ &= \bigcup_{i=1}^{n} (A',B')\circ R_i \\ &= \bigcup_{i=1}^{n} (A',B')\circ (A_i \text{ and } B_i \to C_i) \\ &\triangleq \bigcup_{i=1}^{n} C_i'. \end{aligned}$$

Lemma 2: For the fuzzy conjunctions R_c, R_p, R_{bp}, and R_{dp}, we have

$$\begin{aligned} &(A',B')\circ(A_i \text{ and } B_i \to C_i) \\ &\quad = [A'\circ(A_i \to C_i)] \cap [B'\circ(B_i\to C_i)] \\ &\qquad\qquad \text{if} \quad \mu_{A_i\times B_i} = \mu_{A_i} \wedge \mu_{B_i} \\ &(A',B')\circ(A_i \text{ and } B_i \to C_i) \\ &\quad = [A'\circ(A_i \to C_i)][B'\circ(B_i\to C_i)] \\ &\qquad\qquad \text{if} \quad \mu_{A_i\times B_i} = \mu_{A_i} \cdot \mu_{B_i}. \end{aligned}$$

Proof:

$$\begin{aligned} C_i' &= (A',B')\circ(A_i \text{ and } B_i \to C_i) \\ \mu_{C_i'} &= (\mu_{A'},\mu_{B'})\circ(\mu_{A_i\times B_i}\to\mu_{C_i}) \\ &= (\mu_{A'},\mu_{B'})\circ(\min(\mu_{A_i},\mu_{B_i})\to\mu_{C_i}) \\ &= (\mu_{A'},\mu_{B'})\circ\min[(\mu_{A_i}\to\mu_{C_i}),(\mu_{B_i}\to\mu_{C_i})] \\ &= \sup_{u,v}\min\{[(\mu_{A'},\mu_{B'}), \\ &\qquad \cdot\min[(\mu_{A_i}\to\mu_{C_i}),(\mu_{B_i}\to\mu_{C_i})]\} \\ &= \sup_{u,v}\min\{\min[\mu_{A'},(\mu_{A_i}\to\mu_{C_i})], \\ &\qquad \cdot\min[\mu_{B'},(\mu_{B_i}\to\mu_{C_i})]\} \\ &= \min\{[\mu_{A'}\circ(\mu_{A_i}\to\mu_{C_i})],[\mu_{B'}\circ(\mu_{B_i}\to\mu_{C_i})]\}. \end{aligned}$$

Hence we obtain

$$C_i' = [A'\circ(A_i\to C_i)]\cap[B'\circ(B_i\to C_i)]. \quad \text{Q.E.D.}$$

Let us consider two special cases that follow from the preceding lemma and that play an important role in FLC applications.

Lemma 3: If the inputs are fuzzy singletons, namely, $A' = u_0$, $B' = v_0$, then the results dervied by employing amdani's minimum operation rule R_c and Larsen's oduct operation rule R_p, respectively, may be expressed simply as

$$1)\ \begin{aligned} R_c&: \ \alpha_i^{\wedge} \wedge \mu_{C_i}(w) \\ R_p&: \ \alpha_i^{\wedge}\cdot\mu_{C_i}(w) \end{aligned} \qquad 2)\ \begin{aligned} R_c&: \ \alpha_i^{\cdot} \wedge \mu_{C_i}(w) \\ R_p&: \ \alpha_i^{\cdot}\cdot\mu_{C_i}(w) \end{aligned}$$

where $\alpha_i^{\wedge} = \mu_{A_i}(u_0)\wedge\mu_{B_i}(v_0)$ and $\alpha_i^{\cdot} = \mu_{A_i}(u_0)\cdot\mu_{B_i}(v_0)$.

Proof:

1)

$$\begin{aligned} C_i' &= [A'\circ(A_i\to C_i)]\cap[B'\circ(B_i\to C_i)] \\ \mu_{C_i'} &= \min\{[\mu_0\circ(\mu_{A_i}(u)\to\mu_{C_i}(w))], \\ &\qquad \cdot[v_0\circ(\mu_{B_i}(v)\to\mu_{C_i}(w))]\} \\ &= \min\{[\mu_{A_i}(u_0)\to\mu_{C_i}(w)],[\mu_{B_i}(v_0)\to\mu_{C_i}(w)]\}. \end{aligned}$$

2)

$$\begin{aligned} C_i' &= [A'\circ(A_i\to C_i)]\cdot[B'\circ(B_i\to C_i)] \\ \mu_{C_i'} &= [\mu_0\circ(\mu_{A_i}(u)\to\mu_{C_i}(w))]\cdot[v_0\circ(\mu_{B_i}(v)\to\mu_{C_i}(w))] \\ &= [\mu_{A_i}(u_0)\to\mu_{C_i}(w)]\cdot[\mu_{B_i}(v_0)\to\mu_{C_i}(w)]. \end{aligned}$$

As wil! be seen in following section, the last lemma not only simplifies the process of computation but also provides a graphic interpretation of the fuzzy inference mechanism in the FLC. Turning to the sup–product operator, which is denoted as $\cdot$, we have the following.

Lemma 1': $(A',B')\cdot \bigcup_{i=1}^{n} R_i = \bigcup_{i=1}^{n} (A',B')\cdot R_i$.

Lemma 2': For the fuzzy conjunctions R_c, R_p, R_{bp}, and R_{dp}, we have

$$\begin{aligned} &(A',B')\cdot(A_i \text{ and } B_i \to C_i) \\ &\quad = [A'\cdot(A_i \to C_i)] \cap [B'\cdot(B_i\to C_i)] \\ &\qquad\qquad \text{if} \quad \mu_{A_i\times B_i} = \mu_{A_i} \wedge \mu_{B_i} \\ &(A',B')\cdot(A_i \text{ and } B_i \to C_i) \\ &\quad = [A'\cdot(A_i \to C_i)]\cdot[B'\cdot(B_i\to C_i)] \\ &\qquad\qquad \text{if} \quad \mu_{A_i\times B_i} = \mu_{A_i} \cdot \mu_{B_i}. \end{aligned}$$

Lemma 3: If the inputs are fuzzy singletons, namely, $A' = u_0$, $B' = v_0$, then the results derived by employing Mamdani's minimum operation rule R_c and Larsen's product operation rule R_p, respectively, may be expressed simply as

$$1)\ \begin{aligned} R_c&: \ \alpha_i^{\wedge} \wedge \mu_{C_i}(w) \\ R_p&: \ \alpha_i^{\wedge}\cdot\mu_{C_i}(w) \end{aligned} \qquad 2)\ \begin{aligned} R_c&: \ \alpha_i^{\cdot} \wedge \mu_{C_i}(w) \\ R_p&: \ \alpha_i^{\cdot}\cdot\mu_{C_i}(w) \end{aligned}$$

where $\alpha_i^{\wedge} = \mu_{A_i}(u_0)\wedge\mu_{B_i}(v_0)$ and $\alpha_i^{\cdot} = \mu_{A_i}(u_0)\cdot\mu_{B_i}(v_0)$.

Therefore we can assert that

$$\begin{aligned} R_c&: \quad \mu_{C'} = \bigcup_{i=1}^{n} \alpha_i \wedge \mu_{C_i} \\ R_p&: \quad \mu_{C'} = \bigcup_{i=1}^{n} \alpha_i \cdot \mu_{C_i} \end{aligned}$$

where the weighting factor (firing strength) α_i is a measure of the contribution of the ith rule to the fuzzy control action. The weighting factor in question may be determined by two methods. The first uses the minimum operation in the Cartesian product, which is widely used

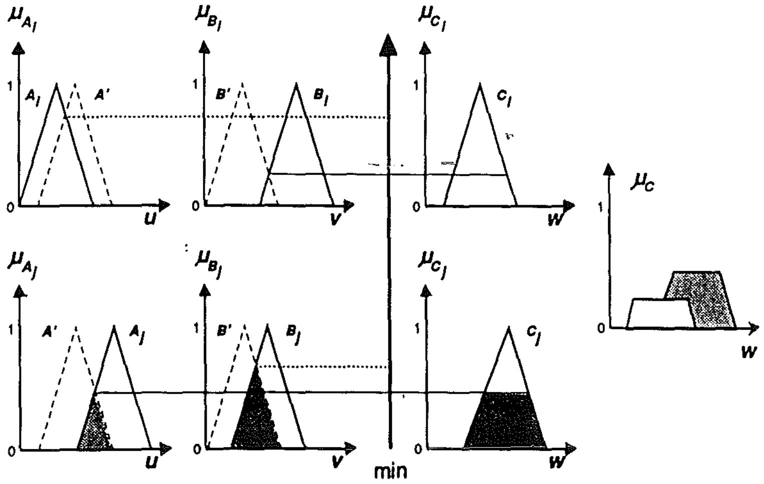

Fig. 10. Graphical interpretation of Lemma 2 under $\alpha^\wedge$ and R_c.

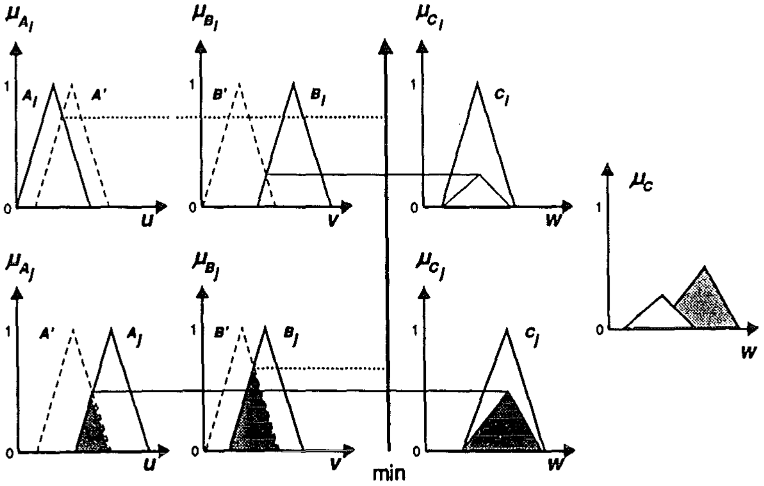

Fig. 11. Graphical interpretation of Lemma 2 under $\alpha^{\cdot}$ and R_p.

in FLC applications. The second employs the algebraic product in the Cartesian product, thus preserving the contribution of each input variable rather than the dominant one only. In this respect, it appears to be a reasonable choice in many FLC applications.

For simplicity, assume that we have two fuzzy control rules, as follows:

R_1: if x is A_1 and y is B_1 then z is C_1,

R_2: if x is A_2 and y is B_2 then z is C_2.

Fig. 10 illustrates a graphic interpretation of Lemma 2 under R_c and $\alpha_i^\wedge$. Fig. 11 shows a graphic interpretation of Lemma 2 under R_p and $\alpha_i^\wedge$.

In on-line processes, the states of a control system play an essential role in control actions. The inputs are usually measured by sensors and are crisp. In some cases it may be expedient to convert the input data into fuzzy sets. In general, however, a crisp value may be treated as a fuzzy singleton. Then the firing strengths α_1 and α_2 of the first and second rules may be expressed as

$$\alpha_1 = \mu_{A_1}(x_0) \wedge \mu_{B_1}(y_0)$$
$$\alpha_2 = \mu_{A_2}(x_0) \wedge \mu_{B_2}(y_0)$$

where $\mu_{A_i}(x_0)$ and $\mu_{B_i}(y_0)$ play the role of the degrees of partial match between the user-supplied data and the data in the rule base. These relations play a central role in the four types of fuzzy reasoning currently employed in FLC applications, and are described in the following.

1) Fuzzy Reasoning of the First Type—Mamdani's Minimum Operation Rule as a Fuzzy Implication Function: Fuzzy reasoning of the first type is associated with the use of Mamdani's minimum operation rule R_c as a fuzzy implication function. In this mode of reasoning, the ith rule leads to the control decision

$$\mu_{C_i'}(w) = \alpha_i \wedge \mu_{C_i}(w)$$

which implies that the membership function μ_C of the inferred consequence C is pointwise given by

$$\mu_C(w) = \mu_{C_1'} \vee \mu_{C_2'}$$
$$= \left[\alpha_1 \wedge \mu_{C_1}(w)\right] \vee \left[\alpha_2 \wedge \mu_{C_2}(w)\right].$$

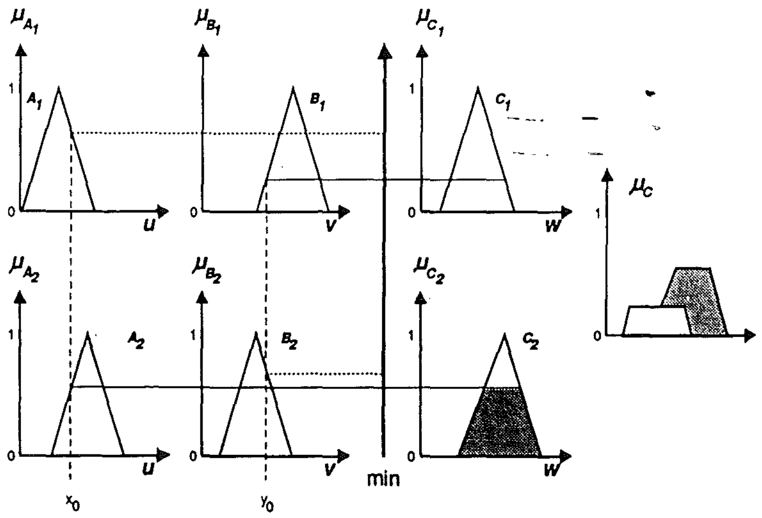

Fig. 12. Diagrammatic representation of fuzzy reasoning 1.

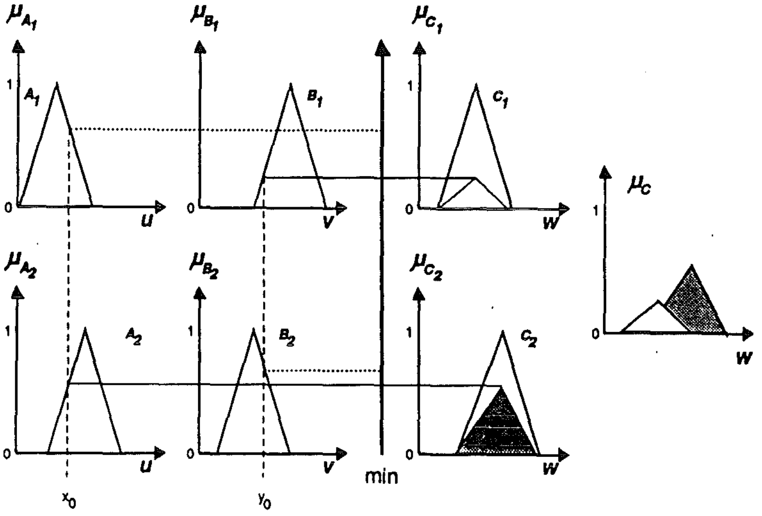

Fig. 13. Diagrammatic representation of fuzzy reasoning 2.

To obtain a deterministic control action, a defuzzification strategy is required, as will be discussed at a later point. The fuzzy reasoning process is illustrated in Fig. 12, which shows a graphic interpretation of Lemma 3 in terms of Mamdani's method R_c.

2) Fuzzy Reasoning of the Second Type—Larsen's Product Operation Rule as a Fuzzy Implication Function: Fuzzy reasoning of the second type is based on the use of Larsen's product operation rule R_p as a fuzzy implication function. In this case, the ith rule leads to the control decision

$$\mu_{C_i'}(w) = \alpha_i \cdot \mu_{C_i}(w).$$

Consequently, the membership function μ_C of the inferred consequence C is pointwise given by

$$\begin{aligned}\mu_C(w) &= \mu_{C_1'} \vee \mu_{C_2'}\\ &= \left[\alpha_1 \cdot \mu_{C_1}(w)\right] \vee \left[\alpha_2 \cdot \mu_{C_2}(w)\right].\end{aligned}$$

From C, a crisp control action can be deduced through the use of a defuzzification operator. The fuzzy reasoning process is illustrated in Fig. 13, which shows a graphic interpretation of Lemma 3 in terms of Larsen's method R_p.

3) Fuzzy Reasoning of the Third Type—Tsukamoto's Method with Linguistic Terms as Monotonic Membership Functions: This method was proposed by Tsukamoto [117]. It is a simplified method based on the fuzzy reasoning of the first type in which the membership functions of fuzzy sets A_i, B_i, and C_i are monotonic. However, in our derivation, A_i and B_i are not required to be monotonic but C_i is.

In Tsukamoto's method, the result inferred from the first rule is α_1 such that $\alpha_1 = C_1(y_1)$. The result inferred from the second rule is α_2 such that $\alpha_2 = C_2(y_2)$. Correspondingly, a crisp control action may be expressed as the weighted combination (Fig. 14)

$$z_0 = \frac{\alpha_1 y_1 + \alpha_2 y_2}{\alpha_1 + \alpha_2}.$$

4) Fuzzy Reasoning of the Fourth Type—The Consequence of a Rule is a Function of Input Linguistic Variables: Fuzzy reasoning of the fourth type employs a modified

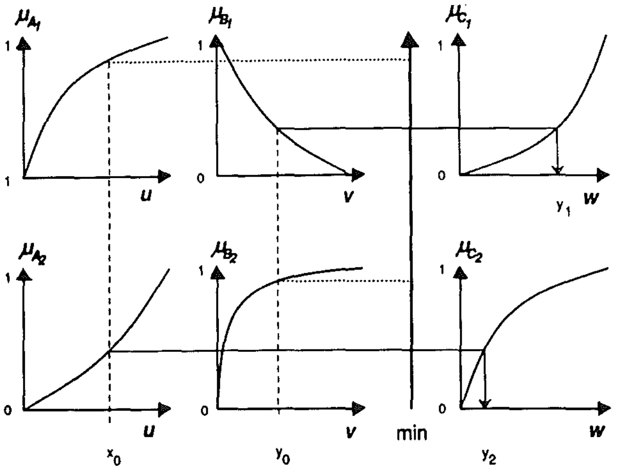

Fig. 14. Diagrammatic representation of fuzzy reasoning 3.

version of state evaluation function. In this mode of reasoning, the ith fuzzy control rule is of the form

$$R_i\text{: if } (x \text{ is } A_i, \cdots \text{ and } y \text{ is } B_i) \text{ then } z = f_i(x, \cdots, y)$$

where $x, \cdots$, y, and z are linguistic variables representing process state variables and the control variable, respectively; $A_i, \cdots, B_i$ are linguistic values of the linguistic variables $x, \cdots, y$ in the universes of discourse $U, \cdots, V$, respectively, with $i = 1, 2, \cdots, n$; and f_i is a function of the process state variables $x, \cdots, y$ defined in the input subspaces.

For simplicity, assume that we have two fuzzy control rules as follows:

$$R_1\text{: if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z = f_1(x, y)$$

$$R_2\text{: if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z = f_2(x, y).$$

The inferred value of the control action from the first rule is $\alpha_1 f_1(x_0, y_0)$. The inferred value of the control action from the second rule is $\alpha_2 f_2(x_0, y_0)$. Correspondingly, a crisp control action is given by

$$z_0 = \frac{\alpha_1 f_1(x_0, y_0) + \alpha_2 f_2(x_0, y_0)}{\alpha_1 + \alpha_2}.$$

This method was proposed by Takagi and Sugeno [103] and has been applied to guide a model car smoothly along a crank-shaped track [98] and to park a car in a garage [97], [99].

II. DEFUZZIFICATION STRATEGIES

Basically, defuzzification is a mapping from a space of fuzzy control actions defined over an output universe of discourse into a space of nonfuzzy (crisp) control actions. It is employed because in many practical applications a crisp control action is required.

A defuzzification strategy is aimed at producing a nonfuzzy control action that best represents the possibility distribution of an inferred fuzzy control action. Unfortunately, there is no systematic procedure for choosing a defuzzification strategy. Zadeh [142] first pointed out this problem and made tentative suggestions for dealing with it. At present, the commonly used strategies may be described as the max criterion, the mean of maximum, and the center of area.

A. The max criterion method

The max criterion produces the point at which the possibility distribution of the control action reaches a maximum value.

B. The Mean of Maximum Method (MOM)

The MOM strategy generates a control action which represents the mean value of all local control actions whose membership functions reach the maximum. More specifically, in the case of a discrete universe, the control action may be expressed as

$$z_0 = \sum_{j=1}^{l} \frac{w_j}{l}$$

where w_j is the support value at which the membership function reaches the maximum value $\mu_z(w_j)$, and l is the number of such support values.

C. The Center of Area Method (COA)

The widely used COA strategy generates the center of gravity of the possibility distribution of a control action. In the case of a discrete universe, this method yields

$$z_0 = \frac{\sum_{j=1}^{n} \mu_z(w_j) \cdot w_j}{\sum_{j=1}^{n} \mu_z(w_j)}$$

where n is the number of quantization levels of the output.

Fig. 15 shows a graphical interpretation of various defuzzification strategies. Braae and Rutherford [5] presented a detailed analysis of various defuzzification strategies (COA, MOM) and concluded that the COA strategy yields superior results (also see [58]). However the MOM strategy yields a better transient performanc while the COA strategy yields a better steady-state per- formance [94]. It should be noted that when the MO strategy is used, the performance of an FLC is similar t that of a multilevel relay system [48], while the CO strategy yields results which are similar to those obtain able with a conventional PI controller [46]. An FLC base on the COA generally yields a lower mean square erro than that based on the MOM [111]. Furthermore, th MOM strategy yields a better performance than the Ma criterion strategy [52].

III. APPLICATIONS AND RECENT DEVELOPMENTS

A. Applications

During the past several years, fuzzy logic has foun numerous applications in fields ranging from finance t earthquake engineering [62]. In particular, fuzzy contr

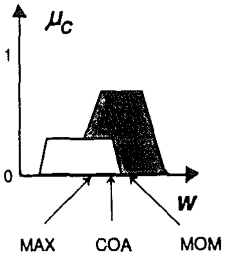

Fig. 15. Diagrammatic representation of various defuzzification strategies.

TABLE VII
FUZZY CONTROL RULES FOR INVERTED PENDULUM BALANCING

		Angle						
		NL	*NM*	*NS*	*ZR*	*PS*	*PM*	*PL*
Change of Angle	*NL*							
	NM							
	NS			*NS*		*ZR*		
	ZR		*NM*		*ZR*		*PM*	
	PS			*ZR*		*PS*		
	PM							
	PL							

has emerged as one of the most active and fruitful areas for research in the application of fuzzy set theory. In many applications, the FLC-based systems have proved to be superior in performance to conventional systems.

Notable applications of FLC include the heat exchange [80], warm water process [47], activated sludge process [113], [35], traffic junction [82], cement kiln [59], [118], aircraft flight control [58], turning process [92], robot control [119], [94], [106], [8], [34], model-car parking and turning [97]–[99], automobile speed control [74], [75], water purification process [127], elevator control [23], automobile transmission control [40], power systems and nuclear reactor control [4], [51], fuzzy memory devices [107], [108], [120], [128], [129], [133], and the fuzzy computer [132]. In this connection, it should be noted that the first successful industrial application of the FLC was the cement kiln control system developed by the Danish cement plant manufacturer F. L. Smidth in 1979. An ingenious application is Sugeno's fuzzy car, which has the capability of learning from examples. More recently, predictive fuzzy control systems have been proposed and successfully applied to automatic train operation systems and automatic container crane operation systems [135]–[139]. In parallel with these developments, a great deal of progress has been made in the design of fuzzy hardware and its use in so-called fuzzy computers [132].

B. Recent Developments

1) Sugeno's Fuzzy Car: One of the most interesting applications of the FLC is the fuzzy car designed by Sugeno. Sugeno's car has successfully followed a crank-shaped track and parked itself in a garage [98]–[99].

The control policy incorporated in Sugeno's car is represented by a set of fuzzy control rules which have the form:

R_i: if x is $A_i, \cdots$ and y is B_i then

$$z = a_0^i + a_1^i x + \cdots + a_n^i y$$

where $x, \cdots$, and y are linguistic variables representing the distances and orientation in relation to the boundaries of the track; $A_i, \cdots$, and B_i are linguistic values of $x, \cdots$, and y; z is the value of the control variable of the ith control rule; and $a_0^i, \cdots$, and a_n^i are the parameters entering in the identification algorithm [103], [99].

The inference mechanism of Sugeno's fuzzy car is based on fuzzy reasoning of the fourth type, with the parameters $a_0^i, \cdots$, and a_n^i identified by training. The training process involves a skilled operator who guides the fuzzy model car under different conditions. In this way, Sugeno's car has the capability of learning from examples.

2) FLC Hardware Systems: A higher-speed FLC hardware system employing fuzzy reasoning of the first type has been proposed by Yamakawa [130], [131]. It is composed of 15 control rule boards and an action interface (i.e., a defuzzifier based on the COA). It can handle fuzzy linguistic rules labeled as *NL, NM, NS, ZR, PS, PM, PL*. The operational speed is approximately 10 mega fuzzy logical inferences per second (FLIPS).

The FLC hardware system has been tested by an application to the stabilization of inverted pendulums mounted on a vehicle. Two pendulums with different parameters were controlled by the same set of fuzzy control rules (Table VII). It is worthy of note that only seven fuzzy control rules achieve this result. Each control rule board and action interface has been integrated to a 40-pin chip.

3) Fuzzy Automatic Train Operation (ATO) Systems: Hitachi Ltd. has developed a fuzzy automatic train operation system (ATO) which has been in use in the Sendai-City subway system in Japan since July 1987. In this system, an object evaluation fuzzy controller predicts the performance of each candidate control command and selects the most likely control command based on a skilled human operator's experience.

More specifically, fuzzy ATO comprises two rule bases which evaluate two major functions of a skilled operator based on the criteria of safety, riding comfort, stop-gap accuracy, traceability of target velocity, energy consumption, and running time. One is constant-speed control (CSC), which starts a train and maintains a prescribed speed. The other is the train automatic stop control (TASC), which regulates a train speed in order to stop at the target position at a station. Each rule base consists of twelve object-evaluation fuzzy control rules. The antecedent of every control rule performs the evaluation of train operation based on safety, riding comfort, stop-gap accuracy, etc. The consequent determines the control action to be taken based on the degree of satisfaction of each criterion. The control action is the value of the train control notch, which is evaluated every 100 ms from the maximal evaluation of each candidate control action, and it takes as a value a discrete number; positive value means "power notch," negative value means "break notch."

The Sendai-City subway system has been demonstrated to be superior in performance to the conventional PID ATO in riding comfort, stop gap accuracy, energy consumption, running time, and robustness [135], [136], [139].

4) Fuzzy Automatic Container Crane Operation (ACO) Systems: In the application of FLC to the automatic operation of container-ship loading cranes, the principal performance criteria are safety, stop-gap accuracy, container sway, and carrying time.

Fuzzy ACO involves two major operations: the trolley operation and the wire rope operation. Each operation comprises two function levels: a decision level and an activation level. Field tests of fuzzy ACO systems with real container cranes have been performed at the port of Kitakyusyu in Japan. The experimental results show that cargo handling ability of Fuzzy ACO by an unskilled operator is more than 30 containers per hour, which is comparable to the performance of a veteran operator. The tests have established that the fuzzy ACO controller has the capability of operating a crane as safely, accurately, and skillfully as a highly experienced human operator [137]–[139].

5) Fuzzy Logic Chips and Fuzzy Computers: The first fuzzy logic chip was designed by Togai and Watanabe at AT&T Bell Laboratories in 1985 [107]. The fuzzy inference chip, which can process 16 rules in parallel, consists of four major parts: a rule-set memory, an inference-processing unit, a controller, and an input–output circuitry. Recently, the rule-set memory has been implemented by a static random access memory (SRAM) to realize a capability for dynamic changes in the rule set. The inference-processing unit is based on the sup–min compositional rule of inference. Preliminary timing tests indicate that the chip can perform approximately 250000 FLIPS at 16-MHz clock. A fuzzy logic accelerator (FLA) based on this chip is currently under development [108], [120]. Furthermore, in March 1989 the Microelectronics Center of North Carolina successfully completed the fabrication of the world's fastest fuzzy logic chip, designed by Watanabe. The full-custom chip comprises 688000 transistors and is capable of making 580000 FLIPS.

In Japan, Yamakawa and Miki realized nine basic fuzzy logic functions by the standard CMOS process in current-mode circuit systems [128]. Later, a rudimentary concept of a fuzzy computer was proposed by Yamakawa and built by OMRON Tateishi Electric Co. Ltd [132]. The Yamakawa-OMRON computer comprises a fuzzy memory, a set of inference engines, a MAX block, a defuzzifier, and a control unit. The fuzzy memory stores linguistic fuzzy information in the form of membership functions. It has a binary RAM, a register, and a membership function generator [128]. A membership function generator (MFG) consists of a PROM, a pass transistor array, and a decoder. Every term in a term set is represented by a binary code and stored in a binary RAM. The corresponding membership functions are generated by the MFG via these binary codes. The inference engine employs MAX and MIN operations, which are implemented by the emitter coupled fuzzy logic gates (ECFL gates) in voltage-mode circuit systems. The linguistic inputs, which are represented by analog voltages distributed on data buses, are fed into each inference engine in parallel. The results inferred from the rules are aggregated by a MAX block, which implements the function of the connective "also" as a union operation, yielding a consequence which is a set of analog voltages distributed on output lines. In the FLC applications, a crisp control command necessitates an auxiliary defuzzifier. In this implementation, a fuzzy computer is capable of processing fuzzy information at the very high speed of approximately 10 mega-FLIPS. It is indeed an important step not only in industrial applications but also in common-sense knowledge processing.

IV. FUTURE STUDIES AND PROBLEMS

In many of its applications, FLC is either designed by domain experts or in close collaboration with domain experts. Knowledge acquisition in FLC applications plays an important role in determining the level of performance of a fuzzy control system. However, domain experts and skilled operators do not structure their decisionmaking in any formal way. As a result, the process of transferring expert knowledge into a usable knowledge base of an FLC is time-consuming and nontrivial. Although fuzzy logic provides an effective tool for linguistic knowledge representation and Zadeh's compositional rule of inference serves as a useful guideline, we are still in need of more efficient and more systematic methods for knowledge acquisition.

An FLC based on the fuzzy model of a process is needed when higher accuracy and reliability are required. However, the fuzzy modeling of a process is still not well understood due to difficulties in modeling the linguistic structure of a process and obtaining operating data in industrial process control [13], [84], [111], [125], [104], [101].

Classical control theory has been well developed and provides an effective tool for mathematical system analysis and design when a precise model of a system is available. In a complementary way, FLC has found many practical applications as a means of replacing a skilled human operator. For further advances, what is needed at this juncture are well-founded procedures for system design. In response to this need, many researchers are engaged in the development of a theory of fuzzy dynamic systems which extends the fundamental notions of state [6], controllability [31], and stability [77], [44], [89], [55].

Another direction of recent exploration is the conception and design of fuzzy systems that have the capability to learn from experience. In this area, a combination of techniques drawn from both fuzzy logic and neural network theory may provide a powerful tool for the design of systems which can emulate the remarkable human ability to learn and adapt to changes in environment.

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REFERENCES

[1] N. Baaklini and E. H. Mamdani, "Prescriptive methods for deriving control policy in a fuzzy-logic controller," *Electron. Lett.*, vol. 11, pp. 625–626, 1975.

[2] J. Baldwin and N. Guild, "Modeling controllers using fuzzy relations," *Kybernetes*, vol. 9, pp. 223–229, 1980.

[3] J. F. Baldwin and B. W. Pilsworth, "Axiomatic approach to implication for approximate reasoning with fuzzy logic," *Fuzzy Sets Syst.*, vol 3, pp. 193–219, 1980.

[4] J. A. Bernard, "Use of rule-based system for process control," *IEEE Contr. Syst. Mag.*, vol. 8, no. 5, pp. 3–13, 1988.

[5] M. Braae and D. A. Rutherford, "Fuzzy relations in a control setting," *Kybernetes*, vol. 7, no. 3, pp. 185–188, 1978.

[6] ____, "Theoretical and linguistic aspects of the fuzzy logic controller," *Automat.*, vol. 15, no 5, pp. 553–577, 1979.

[7] ____, "Section of parameters for a fuzzy logic controller," *Fuzzy Sets Syst.*, vol. 2, no. 3, pp. 185–199, 1979.

[8] K. Ciliz, J. Fei, K. Usluel, and C. Isik, "Pratical aspects of the knowledge-based control of a mobile robot motion," in *Proc. 30th Midwest Symp. on Circuits and Systems*, Syracuse, NY, 1987.

[9] E. Czogala, *Probabilistic Sets in Decision Making and Control.* Cologne: Verlag TUV Rheinland, 1984.

[10] E. Czogala, "Probabilistic fuzzy controller as a generalization of the concept of fuzzy controller," *Fuzzy Sets Syst.*, vol. 26, pp. 215–223, 1988.

[11] E. Czogala and K. Hirota, *Probabilistic Sets: Fuzzy and Stochastic Approach to Decision, Control and Recognition Processes.* Cologne: Verlag TUV Rheinland, 1986.

[12] E. Czogala and W. Pedrycz, "Some problems concerning the construction of algorithms of decision-making in fuzzy systems," *Int. J. Man. Mach. Studies*, vol. 15, pp. 201–221, 1981.

[13] ____, "On identification in fuzzy systems and its applications in control problems," *Fuzzy Sets Syst.*, vol. 6, no. 1, pp. 73–83, 1981.

[14] ____, "Control problems in fuzzy systems," *Fuzzy Sets Syst.*, vol. 7, no. 3, pp. 257–274, 1982.

[15] ____, , "Fuzzy rule generation for fuzzy control," *Cybern. Syst.*, vol. 13, no. 3, pp. 275–293, 1982.

[16] E. Czogala and T. Rawlik, "Modelling of a fuzzy controller with application to the control of biological processes," *Fuzzy Sets Syst.*, vol. 31, pp. 13–22, 1989.

[17] B. Bharathi Devi and V. V. S. Sarma, "Estimation of fuzzy memberships from histograms," *Inform. Sci.*, vol. 35, pp. 43–59, 1985.

[18] D. Dubois and H. Prade, "Fuzzy logics and the generalized modus ponens revisited," *Cybern. Syst.*, vol. 15, pp. 3–4, 1984.

[19] ____, "The generalized modus ponens under sup-min composition—A theoretical study," in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Eds. Amsterdam: North-Holland, 1985, pp. 217–232.

[20] ____, "Unfair coins and necessity measures: Toward a possibilistic interpretation of histograms," *Fuzzy Sets Syst.*, vol. 10, no. 1, pp. 15–20, 1985.

[21] ____, *Possibility Theory: An Approach to Computerized Processing of Uncertainty.* New York: Plenum Press, 1988.

[22] J. Efstathiou, "Rule-based process control using fuzzy logic," in *Approximate Reasoning in Intelligent Systems, Decision and Control*, E. Sanchez and L. A. Zadeh, Eds. New York: Pergamon, 1987, pp. 145–158.

[23] F. Fujitec, "FLEX-8800 series elevator group control system," Fujitec Co., Ltd., Osaka, Japan, 1988.

[24] S. Fukami, M. Mizumoto, and K. Tanaka, "Some considerations of fuzzy conditional inference," *Fuzzy Sets Syst.*, vol. 4, pp. 243–273, 1980.

[25] B. R. Gaines and L. J. Kohout, "The fuzzy decade: A bibliography of fuzzy systems and closely related topics," *Int. J. Man. Mach. Studies*, vol. 9, pp. 1–68, 1977.

[26] S. Gottwald and W. Pedrycz, "Problems of the design of fuzzy controllers," in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Ed. Amsterdam: North-Holland, 1985, pp. 393–405.

[27] B. P. Graham and R. B. Newell, "Fuzzy identification and control of a liquid level rig," *Fuzzy Sets Syst.*, vol. 26, pp. 255–273, 1988.

[28] ____, "Fuzzy adaptive control of a first-order process," *Fuzzy Sets Syst.*, vol. 31, pp. 47–65, 1989.

[29] I. Graham and P. L. Jones, *Expert Systems Knowledge, Uncertainty and Decision.* New York: Chaman and Hall, 1988.

[30] M. M. Gupta and Y. Tsukamoto, "Fuzzy logic controllers—A perspective," in *Proc. Joint Automatic Control Conf.*, San Francisco, Aug. 1980. pp. FA10-C.

[31] M. M. Gupta, G. M. Trojan, and J. B. Kiszka, "Controllability of fuzzy control systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-16, no 4, pp. 576–582, 1986.

[32] K. Hirota, Y. Arai, and S. Hachisu, "Moving mark recognition and moving object manipulation in fuzzy controlled robot," *Cont. Theory Adv. Technol.*, vol. 2, no. 3, pp. 399–418, 1986.

[33] K. Hirota and W. Pedrycz, "Analysis and synthesis of fuzzy systems by the use of fuzzy sets," *Fuzzy Sets Syst.*, vol. 10, no 1, pp. 1–14, 1983.

[34] C. Isik, "Identification and fuzzy rule-based control of a mobile robot motion," in *Proc. IEEE Int. Symposium Intelligent Control*, Philadelphia, PA, 1987.

[35] O. Itoh, K. Gotoh, T. Nakayama, and S. Takamizawa, "Application of fuzzy control to activated sludge process," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July, 1987, pp. 282–285.

[36] A. Jones, A Kaufmann, and H. J. Zimmermann, *Fuzzy Sets Theory and Applications.* Boston: D. Reidel Publishing Company, 1986.

[37] J. Kacprzyk and S. A. Orlovski, *Optimization Models using Fuzzy Sets and Possibility Theory.* Boston: D. Reidel Publishing Company, 1987.

[38] J. Kacprzyk and M. Fedrizzi, *Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making.* New York: Springer-Verlag, 1988.

[39] A. Kandel, *Fuzzy Techniques in Pattern Recognition.* New York: Wiley-Interscience, 1982.

[40] Y. Kasai and Y. Morimoto, "Electronically controlled continuously variable transmission," in *Proc. Int. Congress on Transportation Electronics*, Dearborn, MI, 1988.

[41] A. Kaufmann, *Introduction to Theory of Fuzzy Subsets.* New York: Academic, 1975.

[42] A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic.* New York: Van Nostrand, 1985.

[43] ____, *Fuzzy Mathematical Models in Engineering and Management Science.* Amsterdam: North-Holland, 1988.

[44] M. Kelemen, Fuzziness and structural stability," *Kybernetes*, vol. 10, pp. 67–69, 1981.

[45] W. J. M. Kickert, "Towards an analysis of linguistic modeling," *Fuzzy Sets Syst.*, vol. 2, pp. 293–307, 1979.

[46] ____, "Further analysis and application of fuzzy logic control." Internal Rep. F/WK2/75, Queen Mary College, London, 1975.

[47] W. J. M. Kickert and H. R. Van Nauta Lemke, "Application of a fuzzy controller in a warm water plant," *Automat.*, vol. 12, no. 4, pp. 301–308, 1976.

[48] W. J. M. Kickert and E. H. Mamdani, "Analysis of a fuzzy logic controller," *Fuzzy Sets Syst.*, vol. 1, no. 1, pp. 29–44, 1978.

[49] P. J. King and E. H. Mamdani, "The application of fuzzy control systems to industrial processes," in *IFAC World Congress*, MIT, Boston, 1975.

[50] ____, "The application of fuzzy control systems to industrial processes," *Automat.*, vol. 13, no. 3, pp. 235–242, 1977.

[51] M. Kinoshita, T. Fukuzaki, T. Satoh, and M. Miyake, "An automatic operation method for control rods in BWR plants," in *Proc. Specialists' Meeting on In-Core Instrumentation and Reactor Core Assessment*, Cadarache, France, 1988.

[52] J. B. Kiszka, M. E. Kochanska, and D. S. Sliwinska, "The influence of some parameters on the accuracy of fuzzy model," in

Industrial Applications of Fuzzy Control. M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 187–230.

[53] ____, "The influence of some fuzzy implication operators on the accuracy of a fuzzy model—Part I," *Fuzzy Sets Syst.*, vol. 15, pp. 111–128, 1985.

[54] ____, "The influence of some fuzzy implication operators on the accuracy of a fuzzy model—Part II," *Fuzzy Sets Syst.*, vol. 15, pp. 223–240, 1985.

[55] J. B. Kiszka, M. M. Gupta, and P. N. Nikiforuk, "Energetistic stability of fuzzy dynamic systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 5, pp. 783–792, 1985.

[56] G. J. Klir and T. A. Folger, *Fuzzy Sets, Uncertainty, and Information.* Englewood Cliffs, NJ: Prentice Hall, 1988.

[57] S. V. Komolov, S. P. Makeev, and I. F. Shaknov, "Optimal control of a finite automation with fuzzy constraints and a fuzzy target," *Cybern.*, vol. 16, no. 6, pp. 805–810, 1979.

[58] L. I. Larkin, "A fuzzy logic controller for aircraft flight control," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 87–104.

[59] P. M. Larsen, "Industrial applications of fuzzy logic control," *Int. J. Man Mach. Studies*, vol. 12, no. 1, pp. 3–10, 1980.

[60] E. Lembessis, "Dynamical learning behavior of a rule-based self-organising controller," Ph.D. thesis, Queen Mary College, Univ. of London, 1984.

[61] Y. F. Li and C. C. Lau, "Development of fuzzy algorithms for servo systems," *IEEE Contr. Syst. Mag.*, vol. 9, no. 3, pp. 65–72, 1989.

[62] J. Maiers and Y. S. Sherif, "Applications of fuzzy set theory," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 1, pp. 175–189, 1985.

[63] E. H. Mamdani, "Applications of fuzzy algorithms for simple dynamic plant," *Proc. IEE*, vol. 121, no. 12, pp. 1585–1588, 1974.

[64] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *Int. J. Man Mach. Studies*, vol. 7, no 1, pp. 1–13, 1975.

[65] E. H. Mamdani, "Advances in the linguistic synthesis of fuzzy controllers," *Int. J. Man Mach. Studies*, vol. 8, no. 6, pp. 669–678, 1976.

[66] ____, "Application of fuzzy logic to approximate reasoning using linguistic synthesis," *IEEE Trans. Computer*, vol. C-26, no. 12, pp. 1182–1191, 1977.

[67] ____, "Applications of fuzzy set theory to control systems: a survey," in *Fuzzy Automata and Decision Processes*, M. M. Gupta, G. N. Sardis, and B. R. Gaines, Eds. Amsterdam: North-Holland, 1977, pp. 77–88.

[68] E. H. Mamdani and B. R. Gaines, *Fuzzy Reasoning and its Applications.* London: Academic, 1981.

[69] K. Matsushima and H. Sugiyama, "Human operator's fuzzy model in man-machine system with a nonlinear control object," in *First Int. Conf. on Fuzzy Information Processing*, 1984.

[70] M. Mizumoto, "Note on the arithmetic rule by Zadeh for fuzzy conditional inference," *Cybern. Syst.*, vol. 12, pp. 247–306, 1981.

[71] M. Mizumoto and H. Zimmermann, "Comparison of fuzzy reasoning methods," *Fuzzy Sets and Systems*, vol. 8, pp. 253-283, 1982.

[72] M. Mizumoto,"Comparison of various fuzzy reasoning methods," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 2–7.

[73] ____, "Fuzzy controls under various approximate reasoning methods," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July, 1987, pp. 143–146.

[74] S. Murakami, "Application of fuzzy controller to automobile speed control system," in *Proc. of the IFAC Symp. on Fuzzy Information, Knowledge Representation and Decision Analysis*, Marseille, France, 1983, pp. 43–48.

[75] S. Murakami and M. Maeda, "Application of fuzzy controller to automobile speed control system," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 105–124.

[76] Y. Murayama and T. Terano, "Optimizing control of diesel engine," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 63–72.

[77] C. V. Negoita, "On the stability of fuzzy systems," in *Proc. IEEE Int. Conf. on Cybernetics and Society*, 1978, pp. 936–937.

[78] ____, *Expert Systems and Fuzzy Systems.* Menlo Park: Benjamin/Cummings Publishing Company, 1985.

[79] A. Di Nola and A. G. S. Ventre, *The Mathematics of Fuzzy Systems.* Gesamtherstellung: Verlag TUV Rheinland, 1986.

[80] J. J. Ostergaad, "Fuzzy logic control of a heat exchange process," in *Fuzzy Automata and Decision Processes*, M. M. Gupta, G. N. Saridis, and B. R. Gaines, Eds. Amsterdam: North-Holland, 1977, pp. 285–320.

[81] S. K. Pal and D. K. Dutta Majumder, *Fuzzy Mathematical Approach to Pattern Recognition.* New Delhi: Wiley Eastern Limited, 1986.

[82] C. P. Pappis and E. H. Mamdani, "A fuzzy logic controller for a traffic junction," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-7, no. 10, pp. 707–717, 1977.

[83] W. Pedrycz, "An approach to the analysis of fuzzy systems," *Int. J. Control*, vol. 34, no. 3, pp. 403–421, 1981.

[84] ____, "An identification of fuzzy relational systems," *Fuzzy Sets Syst.*, vol. 13, pp. 153–167, 1984.

[85] ____, "Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data," *Fuzzy Sets Syst.*, vol. 16, pp. 163–175, 1985.

[86] H. Prade and C. V. Negoita, *Fuzzy Logic in Knowledge Engineering.* Gesamtherstellung: Verlag TUV Rheinland, 1986.

[87] T. J. Procyk and E. H. Mamdani, "A linguistic self-organizing process controller," *Automat.*, vol. 15, no. 1, pp. 15–30, 1979.

[88] G. P. Rao and D. A. Rutherford, "Approximate reconstruction of mapping functions from linguistic description in problems of fuzzy logic applied to system control," *Cybernet.*, vol. 12, no. 3, pp. 225–236, 1981.

[89] K. S. Ray and D. D. Majumder, "Application of circle criteria for stability analysis of linear SISO and MIMO systems associated with fuzzy logic controller," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-14, no. 2, pp. 345–349, 1984.

[90] D. A. Rutherford and G. C. Bloore, "The implementation of fuzzy algorithms for control," in *Proc. IEEE*, vol. 64, no. 4, pp. 572–573, 1976.

[91] D. A. Rutherford and G. A. Carter, "A Heuristic adaptive controller for a sinter plant," in *Proc. 2nd IFAC Symp. on Automation in Mining, Mineral and Metal Processing*, Hohannesburg, RSA, Sept. 1976.

[92] Y. Sakai, "A fuzzy controller in turning process automation," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 139–152.

[93] E. Sanchez and L. A. Zadeh, *Approximate Reasoning in Intelligent Systems, Decision and Control.* New York: Pergamnon, 1988.

[94] E. M. Scharf and N. J. Mandic, "The application of a fuzzy controller to the control of a multi-degree-freedom robot arm," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 41–62.

[95] S. Shao, "Fuzzy self-organizing controller and its application for dynamic processes," *Fuzzy Sets Syst.*, vol. 26, pp. 151–164, 1988.

[96] M. S. Stachowicz and M. E. Kochanska, "Fuzzy modeling of the process," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 86–89.

[97] M. Sugeno and K. Murakami, "Fuzzy parking control of model car," in *23rd IEEE Conf. on Decision and Control*, Las Vegas, 1984.

[98] M. Sugeno and M. Nishida, "Fuzzy control of model car," *Fuzzy Sets Syst.*, vol. 16, pp. 103–113, 1985.

[99] M. Sugeno and K. Murakami, "An experimental study on fuzzy parking control using a model car," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 125–138.

[100] M. Sugeno, "An introductory survey of fuzzy control," *Inform. Sci.*, vol. 36, pp. 59–83, 1985.

[101] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets Syst.*, vol. 28, no. 1, pp. 15–33, 1988.

[102] K. Sugiyama, "Analysis and synthesis of the rule based self-organising controller," Ph.D thesis, Queen Mary College, Univ. of London, 1986.

[103] T. Takagi and M. Sugeno, "Derivation of fuzzy control rules from human operator's control actions," in *Proc. of the IFAC Symp. on Fuzzy Information, Knowledge Representation and Decision Analysis*, Marseilles, France, July 1983, pp. 55–60.

[104] ____, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-15, no. 1, pp. 116–132, 1985.

[105] K. L. Tang and R. J. Mulholland, "Comparing fuzzy logic with classical controller designs," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-17, no. 6, pp. 1085–1087, 1987.

[106] R. Tanscheit and E. M. Scharf, "Experiments with the use of a

rule-based self-organising controller for robotics applications," *Fuzzy Sets Syst.*, vol. 26, pp. 195–214, 1988.

M. Togai and H. Watanabe, "Expert system on a chip: An engine for real-time approximate reasoning," *IEEE Expert Syst. Mag.*, vol. 1, pp. 55–62, 1986.

[108] M. Togai and S. Chiu, "A fuzzy accelerator and a programming environment for real-time fuzzy control," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 147–151.

[109] R. M. Tong, "Analysis of fuzzy control algorithms using the relation matrix," *Int. J. Man Mach. Studies*, vol. 8, no. 6, pp. 679–686, 1976.

[110] ____, "A control engineering review of fuzzy systems," *Automa.*, vol. 13, no. 6, pp. 559–569, 1977.

[111] ____, "Synthesis of fuzzy models for industrial processes," *Int. Gen. Syst.*, vol. 4, pp. 143–162, 1978.

[112] ____, "The evaluation of fuzzy models derived from experimental data," *Fuzzy Sets Syst.*, vol. 4, pp. 1–12, 1980.

[113] R. M. Tong, M. B. Beck, and A. Latten, "Fuzzy control of the activated sludge wastewater treatment process," *Automat.*, vol. 16, no. 6, pp. 695–701, 1980.

[114] R. M. Tong, "A retrospective view of fuzzy control systems," *Fuzzy Sets Syst.*, vol. 14, pp. 199–210, 1984.

[115] ____, "An annotated bibliography of fuzzy control," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 249–269.

[116] E. Trillas and L. Valverde, "On mode and implication in approximate reasoning," in *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Eds. Amsterdam: North-Holland, 1985, pp. 157–166.

[117] Y. Tsukamoto, "An approach to fuzzy reasoning method," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragade, and R. R. Yager, Eds. Amsterdam: North-Holland, 1979.

[118] I. G. Umbers and P. J. King, "An analysis of human-decision making in cement kiln control and the implications for automation," *Int. J. Man Mach. Studies*, vol. 12, no. 1, pp. 11–23, 1980.

] M. Uragami, M. Mizumoto, and K. Tananka, "Fuzzy robot controls," *Cybern.*, vol. 6, pp. 39–64, 1976.

[120] H. Watanabe and W. Dettloff, "Reconfigurable fuzzy logic processor: A full custom digital VLSI," in *Int. Workshop on Fuzzy System Applications*, Iizuka, Japan, Aug. 1988, pp. 49–50.

[121] D. Willaeys, "Optimal control of fuzzy systems," in *Proc. Int. Congress on Applied Systems Research and Cybernetics*, Acapulco, Dec. 1980.

[122] ____, "Decision making with possibilistic dynamic systems," in *Proc. 20th IEEE Conf. on Decision and Control*, San Diego, Dec. 1981.

[123] D. Willaeys and N. Malvache, "Use of fuzzy model for process control," in *Proc. IEEE Int. Conf. on Cybernetics and Society*, 1978, pp. 942–946.

[124] ____, "Contribution of the fuzzy sets theory to man-machine system," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragade, and R. R. Yager, Eds. Amsterdam: North-Holland, 1979, pp. 481–500.

[125] C. W. Xu and Y. Zailu, "Fuzzy model identification and self-learning for dynamic systems," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-17, no. 4, pp. 683–689, 1987.

[126] R. R. Yager, S. Ovchinnikov, R. M. Tong, and H. T. Nguyen, *Fuzzy Sets and Applications: Selected Papers by L. A. Zadeh*. New York: Wiley-Interscience, 1987.

[127] O. Yagishita, O. Itoh, and M. Sugeno, "Application of fuzzy reasoning to the water purification process," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 19–40.

[128] T. Yamakawa and T. Miki, "The current mode fuzzy logic integrated circuits fabricated by the standard CMOS process," *IEEE Trans. Computer*, vol. C-35, no. 2, pp. 161–167, 1986.

[129] T. Yamakawa and K. Sasaki, "Fuzzy memory device," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 551–555.

[130] T. Yamakawa, "High speed fuzzy controller hardware system," in *Proc. 2nd Fuzzy System Symp.*, Japan, 1986, pp. 122–130.

[131] ____, "Fuzzy controller hardware system," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987.

[132] ____, "A simple fuzzy computer hardware system employing min and max operations—A challenge to 6th generation computer," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987.

[133] T. Yamakawa, "Fuzzy microprocessors—Rule chip and defuzzifier chip," in *Int. Workshop on Fuzzy System Applications*, Iizuka, Japan, Aug. 1988, pp. 51–52.

[134] T. Yamazaki and M. Sugeno, "A microprocessor based fuzzy controller for industrial purposes," in *Industrial Applications of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 231–240.

[135] S. Yasunobu, S. Miyamoto, and H. Ihara, "Fuzzy control for automatic train operation system," in *Proc. 4th IFAC/IFIP/IFORS Int. Congress on Control in Transportation Systems*, Baden-Baden, April 1983.

[136] S. Yasunobu and S. Miyamoto, "Automatic train operation by predictive fuzzy control," in *Industrial Application of Fuzzy Control*, M. Sugeno, Ed. Amsterdam: North-Holland, 1985, pp. 1–18.

[137] S. Yasunobu and T. Hasegawa, "Evaluation of an automatic container crane operation system based on predictive fuzzy control," *Control Theory Adv. Technol.*, vol. 2, no. 3, pp. 419–432, 1986.

[138] ____, "Predictive fuzzy control and its application for automatic container crane operation system," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 349–352.

[139] S. Yasunobu, S. Sekino, and T. Hasegawa, "Automatic train operation and automatic crane operation systems based on predictive fuzzy control," in *Proc. 2nd IFSA Congress*, Tokyo, Japan, July 1987, pp. 835–838.

[140] M. Yonekara, "The application of fuzzy sets theory to the temperature of box annealing furnaces using simulation techniques," in *Proc. 8th IFAC World Congress*, Tokyo, 1981.

[141] L. A. Zadeh, "Fuzzy sets," *Informat. Control*, vol. 8, pp. 338–353, 1965.

[142] ____, "Fuzzy algorithm," *Informat. Control*, vol. 12, pp. 94–102, 1968.

[143] ____, "Toward a theory of fuzzy systems," in *Aspects of Network and System Theory*, R. E. Kalman and N. DeClaris, Ed. New York: Holt, Rinehart and Winston, 1971, pp. 469–490.

[144] ____, "Similarity relations and fuzzy orderings," *Informat. Sci.*, vol. 3, pp. 177–200, 1971.

[145] ____, "A rationale for fuzzy control," *Trans. ASME, J. Dynam. Syst. Measur. Control*, vol. 94, pp. 3–4, 1972.

[146] ____, "Outline of a new approach to the analysis complex systems and decision processes," *IEEE Trans. Syst. Man Cybern.*, vol. SMC-3, pp. 28–44, 1973.

[147] ____, "The concept of a linguistic variable and its application to approximate reasoning I, II, III," *Informat. Sci.*, vol. 8, pp. 199–251, pp. 301–357; vol. 9, pp. 43–80, 1975.

[148] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*. Boston: Kluwer Academic Publishers, 1985.

[149] H. J. Zimmermann, *Fuzzy Sets, Decision Making, and Expert Systems*. Boston: Kluwer Academic Publishers, 1987.

[150] C. C. Lee, "Fuzzy logic in control systems: Fuzzy logic controller —Part I," *IEEE Trans. Syst. Man Cybern.*, vol. 20, no. 2, pp. 404–418, Mar./Apr. 1990.

Chuen Chien Lee (S'88) for a biography and photograph, please see page 418 of this TRANSACTIONS.

QUICK LOOK

EDITED BY SHERRIE VAN TYLE

MARKET FACTS

Simulating human reasoning could prove quite lucrative for companies working with neural networks and fuzzy logic. Market Intelligence Research Corp. expects total revenues for neural networks and fuzzy logic combined to grow from $300 million last year to $10 billion by 1998. MIRC forecasts compound annual revenue growth of about 65% in its report, Imitating Human Reasoning: the Viability and Commercialization of Neural Networks and Fuzzy Logic.

Fuzzy logic is intended to develop multivalued rather than binary logic than can simulate human response to continuous rather than discrete choices. Neural networks focus on simulating the high connectivity between the many cells making up the human brain. Of the two, fuzzy logic will grow more rapidly until mid-decade, when neural networks will recapture the lead in growth rate.

Japan is prevailing in fuzzy logic technology, having applied it in at le[illegible]00 applications. In consumer electronics, some video cameras us[illegible]zy logic for focusing. In neural networks, Japan is expected to be a strong competitor with the U. S. For its part, Europe is playing catch-up, but major players Philips, Siemens, and Thomson are moving into both areas.

Standard software and ICs are expected to displace engineering development tools and customer applications as product segments. Neural networks are working their way into financial and industrial environments; in the U. S., the military continues to fund neural network applications. In fuzzy logic, industrial and automotive applications are predicted to overtake the consumer electronics segment.

STRONG GROWTH FOR NEURAL NETWORKS AND FUZZY LOGIC

Year	World revenues ($ million)	Revenue growth rate (percent)
1991	301.3	69.6
1992	580.4	92.6
1993	1,208.7	108.2
1994	2,539.5	110.1
1995	4,643.2	82.8
1996	6,461.6	39.2
1997	7,996.0	23.7
[illegible]998	9,915.0	24.0

[illegible]pound annual growth rate (1991-1998): 64.7%

Note: All figures are rounded.
Source: Market Intelligence Research Corp.

HOT PC PRODUCTS

Most 486SX-based PCs were built with a vacant upgrade socket. Until now, there was no upgrade part to fill that vacancy. The OverDrive processors from Intel, which fit into the empty socket, increase performance up to 70%. The part is currently available in two versions, one for 16- and 20-MHz systems and one for 25-MHz systems. Unlike a math coprocessor, the part improves both floating-point and integer performance on all DOS, Windows, OS/2, and Unix applications. The processor is based on the Intel's DX2 speed-doubling technology, where the internal clock rate is twice that of the external rate.

In most cases, however, installing the processor doesn't require any modifications to the computer. The company says that users should be able to install the OverDrive processor in five minutes. The 16/20-MHz part sells for $549 and the 25-MHz version costs $699. A 486DX part will be available in the fall and a DX2 part should appear in early 1993. Contact Intel Crop., 3065 Bowers Ave., Santa Clara, CA 95051; (800) 538-3373. CIRCLE 481

A kit that enables engineers to implement fuzzy logic on Motorola microcontrollers has a version of FIDE, a fuzzy inference development environment Motorola developed with Aptronix. The basic kit, for $195, has a computer-based course that teaches users how to apply fuzzy logic to their applications, an introductory version of FIDE, related software, and documentation. For board-level, in-circuit emulation, a $600 kit includes an M68HC05EVM emulator (through August). Users need a PC AT or compatible with one floppy drive, a 40-Mbyte hard drive, VGA monitor, DOS 3.30 with Windows 3.0, though DOS 5.0 is recommended. Contact Motorola's Microprocessor and Memory Technologies Group, 6501 William Cannon Dr. W, Austin, TX 78735-8598. CIRCLE 482

MINC has reduced the price of its DOS-based PLDesigner System 200 to $495 (from $1,950) and its System 300 to $795 (from $2,950) through August. The System 200, though an entry-level system, has MINC's high-level Design Synthesis Language, functional simulation, and automatic device selection and device fitting. System 300 can implement multiple-device designs. Optional interfaces link the software to PC-based schematic capture systems. Contact MINC, 6755 Earl Dr., Colorado Springs, CO 80918; (719) 590-1155. CIRCLE 483

Along similar lines, Actel Corp. has cut the price by one-third of its Action Logic System (ALS) Release 2.1. The system is used to design and program the company's Act 1 and 2 field-programmable gate arrays. Release 2.1 is a complete FPGA design, debugging, and programming system for the Act 1 and 2 devices. The system supplies automatic placement and routing for the devices, along with in-circuit diagnostics, minimizing design verification. For PCs, Act 1 goes for $1995, Act 2 for $3495. For Sun systems, Act 1 goes for $3995, Act 2 for $5495. CIRCLE 484

WHAT'S NEW

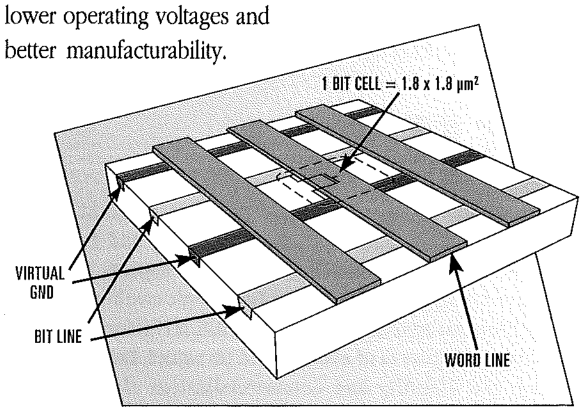
FROM THE LARGEST MROM TO THE FASTEST FIFO, COUNT ON SHARP TO DELIVER.

The pursuit of greater densities — and new applications — drives Sharp just as it does system designers.

The new LH543620, for example, is a 1024 x 36 unidirectional FIFO with the most fully synchronous set of features available, including five programmable flags, independently synchronized operation of input/output ports, and duplicate enables at each end. Full word-width and fast cycle times qualify the LH543620 for even the most high-end instrumentation, DSP and data communications system applications.

Sharp is now proud to offer the world's largest Mask ROM — the 32 Mb LH5332000 — with a 200 ns access time in either a X 8 or X 16 configuration. It's one in a new line of MROMs using Sharp's NOR flat-cell technology to achieve faster speeds, higher densities, lower operating voltages and better manufacturability.

Sharp is the longtime leader in MROM production, worldwide, with an amazing 350 million chips sold since 1979.



IMAGINATION AT WORK

SHARP AND INTEL FORGE FLASH MEMORY PARTNERSHIP.

News of an historic alliance between Sharp and Intel, announced in March, focused attention on Sharp's Fukuyama plant in Hiroshima, Japan, where Intel's flash memory products will be manufactured.

Fukuyama's CIM (Computer Integrated Manufacturing) system is the industry's most advanced, using software designed at Sharp's R&D hub in Camas, Washington, to enable a super-clean manufacturing and quality control environment virtually untouched by human hands. The use of robotics for submicron-level, ultra-high-precision processing of electronic components improves yields via Class 1 clean room standards.

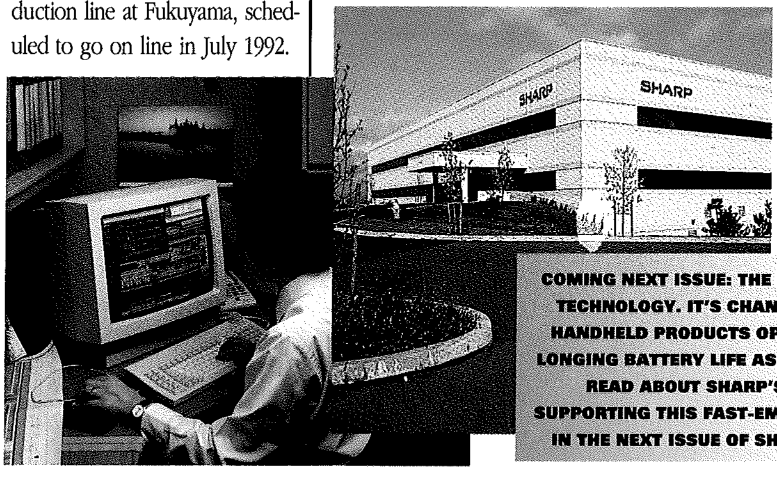
Intel will also benefit from Sharp's $800 million investment in a third, 8″ wafer, .5 micron feature size semiconductor production line at Fukuyama, scheduled to go on line in July 1992.

SHARP IN THE U.S.A.

IC DESIGN IN WASHINGTON BENEFITS U.S MANUFACTURERS.

From its new production and research facility in Camas, Washington, Sharp has the ability to promptly develop application-specific ICs and new processing technologies to match the unique requirements of North American manufacturers.

Why Camas? A supportive business climate, skilled labor pool and geophysical advantages — including excellent ground water quality, essential for a future waferfab — all made the Southwest Washington site a natural.



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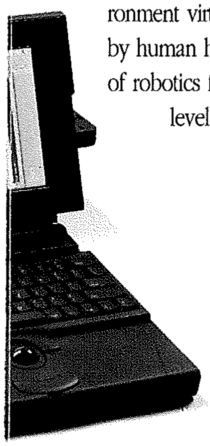

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July 7, 1991

TO: TIE Office, TIE Students & Campus Students of EE595.
FROM: Robert J. Marks II
SUBJECT: Course Project

EE595 PROJECT

You are to read one or more papers in fuzzy systems from a scholarly journal or conference Proceedings.* Two "short" papers (such as *letters to the editor)* will be considered as a single paper. Your knowledge of the paper will be tested in two ways:

1. A one page type written summary of the paper

2. A short oral presentation before the class.

A complete xerox copy of the paper you wish to present will be due in class on ________________________. Those who hand in the sheet late will, at minimum, be chosen to go first. This sheet will be stapled to the paper copy. Please fill out the following information:

your name ____

author(s)____

paper title____

Journal name____

volume ________. pages _____-______. year ___________

In case of duplication of papers, one entry will be asked to change. If no consensus can be reached, this will be determined by a flip of the coin.

Immediately prior to your presentation, be prepared to give the instructor a copy of of you summary stapled to a complete copy of the paper. For both the written and oral presentations, you will be graded on the clarity of your presentation.

Suitable journals include **IEEE Transactions on Systems, Man & Cybernetics, IEEE Transactions on Neural Networks, IEEE Transactions on Patern Analysis & Machine Intelligence, Fuzzy Sets & Systems** and, **The International Journal of Approximate Reasoning.** Magazines, such as **Expert** and **IEEE Spectrum** are not suitable. Neither are lay publications like **Scientific American** and **Omni.** Paper collection volumes, such as **Fuzzy Models for Pattern Recognition** (Bezdek & Pal, IEEE Press, 1992) are excellent sources of classic papers. Many conference records, such as **Proceedings of '92 FUZZ-IEEE** (IEEE Press) contain suitable papers.

---

* Innovative projects concerning fuzzy systems may be proposed in lieu of a paper review.

Points for a Good Oral Presentation

1. There is no substitute for experience. Practice your presentation with a critical friend.

2. Be enthusiastic. To be enthusiastic, *act* enthusiastic. Modulate your voice. Smile.

3. A presentation that runs overtime is a bad presentation. Except for rare exceptions, no one listens to you after your time has expired. Their attention is on the clock and other things. If there is to much material to present, only present the most important points.

4. Present concepts and not a lot a math. Communicate with English - not equations. *Do not* ***read*** *equations.*

5. Use your overhead transparencies (or slides) as your notes. Don't read your slides. Don't read your presentation.

6. Any good oral presentation has three parts:
    (a) *Introduction:* (tell them what you're going to tell them.)
    (b) *Body*: (tell them.)
    (c) *Conclusion*: (tell them what you told them.)

Fuzzy Models for Pattern Recognition

Methods That Search for Structures in Data

Edited by

James C. Bezdek
Division of Computer Science
University of West Florida

Sankar K. Pal
Electronics and Communications Science Unit
Indian Statistical Institute, Calcutta

IEEE
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A Selected Reprint Volume
IEEE Neural Networks Council, *Sponsor*

The Institute of Electrical and Electronics Engineers, Inc., New York

Chapter 3 Classifier Design and Feature Analysis 227

Chapter 4 Image Processing and Machine Vision 327

Project Description for EE499 or EE599

Robert J. Marks II

There are six tapes available from IEEE on Fuzzy Systems.

- Lotfi Zadeh, "Advanced Concepts and Structures"
- Enrique Ruspini, "Introduction to Fuzzy Set Theory and Fuzzy Logic"
- James C. Bezdek, "Fuzzy Logic and Neural Networks for Pattern Recognition"
- James Keller, "Fuzzy Logic and Neural Networks for Computer Vision"
- Hamid R. Berenji, "Fuzzy Logic and Neural Networks for Control Systems"
- Piero Bonissone, "Information Processing with Fuzzy Logic"

For one credit, choose five of these tapes. For each tape, write a review of the presentation and contents. The review should read similar to a book review. There is no specified length of the review. It should be long enough to summarize the contents and provide an overall critique.

The tapes will be available from Ruth Wagner Bennett for two night check out. VCR's are available in the undergraduate library.

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Fuzzy Control Systems: Clear Advantages

Michael Reinfrank

Since 1987 the Japanese city of Sendai has had a driverless subway system that is automatically operated by a so-called fuzzy controller. Whereas in Europe the term "fuzzy" has long aroused negative associations, the Japanese are increasingly embracing the concept and applying it. For instance, fuzzy logic can decide on the optimum time for a car to shift gears, can manage the amount of suction needed by a vacuum cleaner, and can even limit subject movement in video cameras. But now the fuzzy wave has also reached Europe.

Fuzzy Control

What is a fuzzy controller? As far as the operation of a subway system is concerned, the problem can be simplified as follows: the positions of accelerator lever and brake lever must be determined on the basis of available measured data (e.g. current speed, position) and desired targets (e.g. required speed curve). Basically, there are three possible ways of achieving this (Fig. 1). The most widespread method is that of manual operation, i.e. the translation of measured data and target requirements into acceleration and braking actions by the driver. If, however, we wish to automate this operation (one possible way of increasing the frequency of trains in local public transport), the classical approach is based on the following principle: mathematical models are used to provide as accurate a description as possible of the technical process controlled by the driver, and this model is then used as the basis for algorithmic methods, such as so-called PID controllers. Conversely, with fuzzy control, it is not the technical system that is modeled, but the manner in which a human process controller acts, i.e. how the driver drives the train.

But how is a subway train driven? Interviews with drivers and technicians result in the formulation of rules such as the following: If the train is a short distance from the station and is traveling at average speed, then an average braking force is required.

A central problem in this respect is the term "average speed," which must be described in formal terms so that such a rule can be processed in a computer. The first possible solution to this problem is shown in Figure 2: the normal speed range of a subway train is broken down into sections in each of which a clear definition is made: yes, 40 km/h is an average speed or no, 39 km/h is not an average speed. Such a solution entails two problems. No subway driver or technician is able with certainty to draw a precise dividing line between what is and what is not an average speed. Even if such unambiguously defined limits were available, controls based on these would result in a jerky ride at the points of transition between one speed range and the next, since the above rule, for example, is not applied at all at 39 km/h but is wholly enforced at 40 km/h.

This is where fuzzy logic and fuzzy control enter the picture. Such systems make it possible to produce a gradual transition in speed, as shown in Figure 2. There are speed ranges in which the question "Is this an average speed?" can be clearly answered with yes or no; the transitions between ranges, however, are fluid or fuzzy. A speed of 40 km/h corresponds only to a certain extent to a subway driver's concept of an average speed – and only to that extent will a rule responding to such a speed be satisfied and applied. It is important to note that fuzzy control does not necessarily have anything to do with fuzzy data, but with fuzzy control concepts used in the processing of data – of both the fuzzy and non-fuzzy kind.

Typically, a fuzzy control consists of 20 to 100 such rules that are run through in a loop. Measured data and reference variables are inputted into the control at defined intervals; the output from the control comprises control actions or manipulated variables derived using these rules. Consequently, a fuzzy controller is a real-time expert system used in process automation that employs fuzzy logic in order to represent qualitative variables. Both the gradual decision-making functions and the rules and their execution are coupled to very elementary operations, which provides the basis for specific software and hardware support (fuzzy chips), and thus permits efficient, real-time-capable solutions. Considering fuzzy controllers as real-time expert systems, their relation to neuronal networks is of particular interest. Both systems are based on the same principle: they attempt to model human thought processes and, in particular, the soft decisions that occur in such

Dr. Michael Reinfrank,
Siemens AG,
Corporate Research and
Development,
Munich, Germany

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Fuzzy Inference Engines

Robert J. Marks II

1 Inference Engines

1.1 Introduction

The *fuzzy inference engine* is the foundation of most fuzzy expert systems and control systems. From a linguistic description of cause and effect of a process, a fuzzy inference engine can be designed to emulate the process.

1.2 Fuzzy If-Then Rules

Cause and effect statements of a process are stated through if-then rules. Consider the following pedigogical example wherein the success of an undergraduate as a graduate student is inferred through consideration of their undergraduate grade point averages (GPA's) and their performance on the GRE analytic test.

- *IF* an undergraduate's GPA is **high** *AND* their GRE score is **high**, *THEN* an undergraduate student will make an **excellent** graduate student,
- *OR, IF*
  - their GPA is **high** *AND* their GRE score is **fair**,
  - *OR* their GPA is **fair** *AND* their GRE score is **high**.

  THEN an undergraduate student will make a **good** graduate student,
- *OR IF* their GPA is **fair** *AND* their GRE score is **fair**, *THEN* an undergraduate student will make an **average** graduate student,
- *OR, OTHERWISE*, the undergraduate student will make a **poor** graduate student,

Note, first, the operations of *IF, THEN, AND* and *OR*. Each can be interpreted in a fuzzy sense. The fuzzy linguistic variables are written in **bold**. These rules can be simplified using the following linguistic variable abbreviations.

A for **average**,
E for **excellent**
F for **fair**
G for **good**
H for **high**,
L for **low**,
P for **poor**

The If-Then Rules can then be written as

If	GPA is H	And	GRE is H,	Then E
Or, If	[GPA is F	And	GRE is H,	
	Or GPA is H	And	GRE is F],	Then G
Or, If	GPA is F	And	GRE is F,	Then A
Or, If	[GPA is P			
			OR GRE is P],	Then P

The *If* portions of these statements are referred to as *antecedents*. The *Then* portions are the *consequents*.

1.2.1 Fuzzy Numerical Interpretation of the Antecedent

The first step in building the fuzzy inference engine is quantification of the linguistic variables using fuzzy membership functions. Consider again our example about students. Possible membership functions for low, high and very high are shown at the top of Figure 1. The 'low' membership function is denoted by $\mu_{\text{L-GRE}}(S)$ where S is the GRE score. Similarly, the 'fair' and 'high' membership functions are $\mu_{\text{F-GRE}}(S)$ and $\mu_{\text{H-GRE}}(S)$. The midpoint of 'fair' is at a score of 700. The higher the score, the greater the membership in 'excellent' scores.

Similarly, three membership fuctions for low, fair and high undergraduate GPA's is shown in the center of Figure 1. They are, repectively, $\mu_{\text{L-GPA}}(G)$, $\mu_{\text{F-GPA}}(G)$ and $\mu_{\text{H-GPA}}(G)$ where G is the GPA.

Given the GPA and GRE score of a student, the definitions of each of the antecedents can be evaluated using these membership definitions. To illustrate, suppose the GRE score of Student X is 720. With reference to the top of Figure 2, the following GRE membership functions are ascertained.

$$\mu_{\text{L-GRE}}(720) = 0$$

$$\mu_{\text{F-GRE}}(720) = 0.8$$

$$\mu_{\text{H-GRE}}(720) = 0.2$$

If Student X has an undergraduate GPA of 3.7, the corresponding membership functions have values of

$$\mu_{\text{L-GPA}}(3.7) = 0$$

$$\mu_{\text{F-GPA}}(3.7) = 0.6,$$

$$\mu_{\text{H-GPA}}(3.7) = 0.4.$$

Assume the minimum operation if used for the fuzzy 'and'. (Other fuzzy intersetion and union operations can also be used.) Then the the composite membership for the antecedent, GPA is H And GRE is H, is

$$\begin{aligned}\mu_{\text{GPA is H And GRE is H}} &= \min[\mu_{\text{H-GPA}}(3.7), \mu_{\text{H-GRE}}(720) \\ &= \min(0.4, 0.2) \\ &= 0.2\end{aligned}$$

The consequent for a G graduate student is a bit longer

$$\begin{aligned}\mu_{\text{[GPA is F And GRE is H] Or [GPA is H And GRE is F]}} &= \max[\min(0.6, 0.2), \min(0.4, 0.8)] \\ &= 0.4\end{aligned}$$

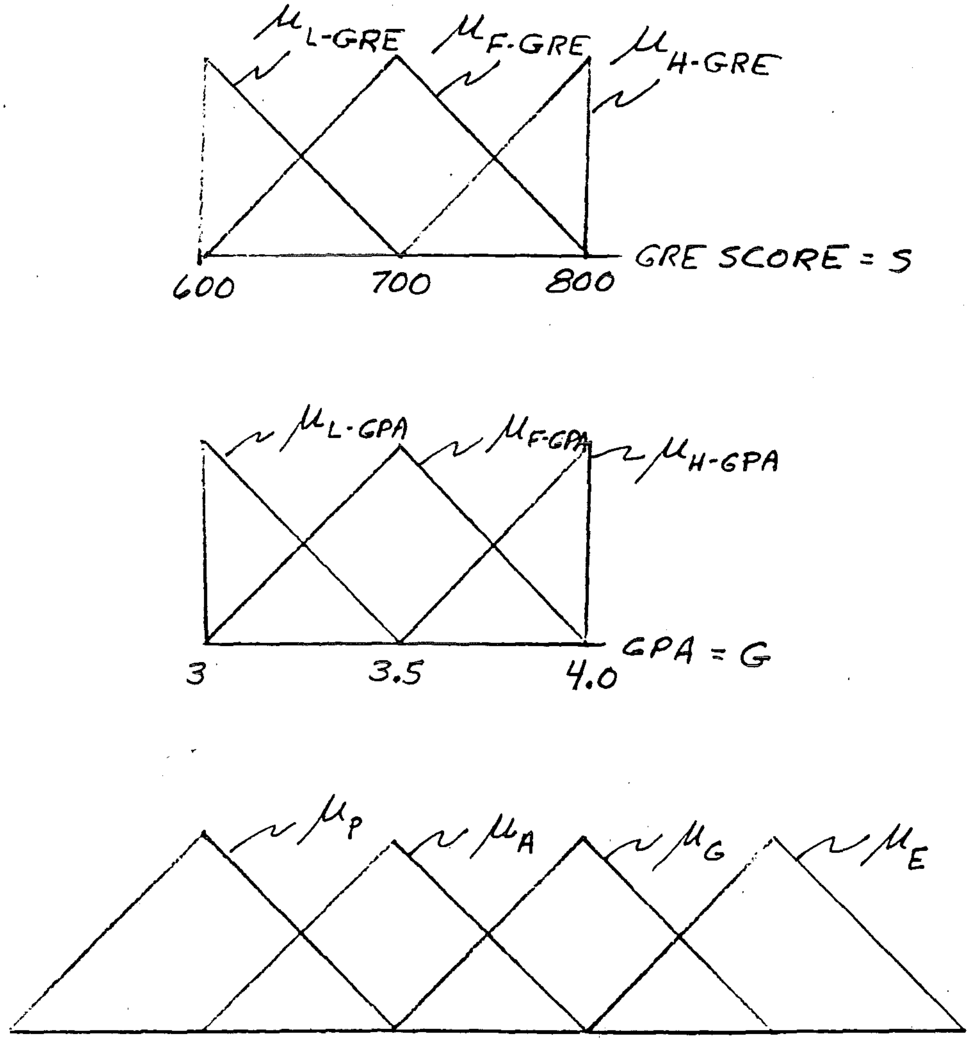

Figure 1: (top) Fuzzy membership functions for L, F and H scores for the analytic GRE test; (center) fuzzy membership functions for L, F and H GPA's; (bottom) fuzzy membership functions for P, A, G and E prospects for graduate student success.

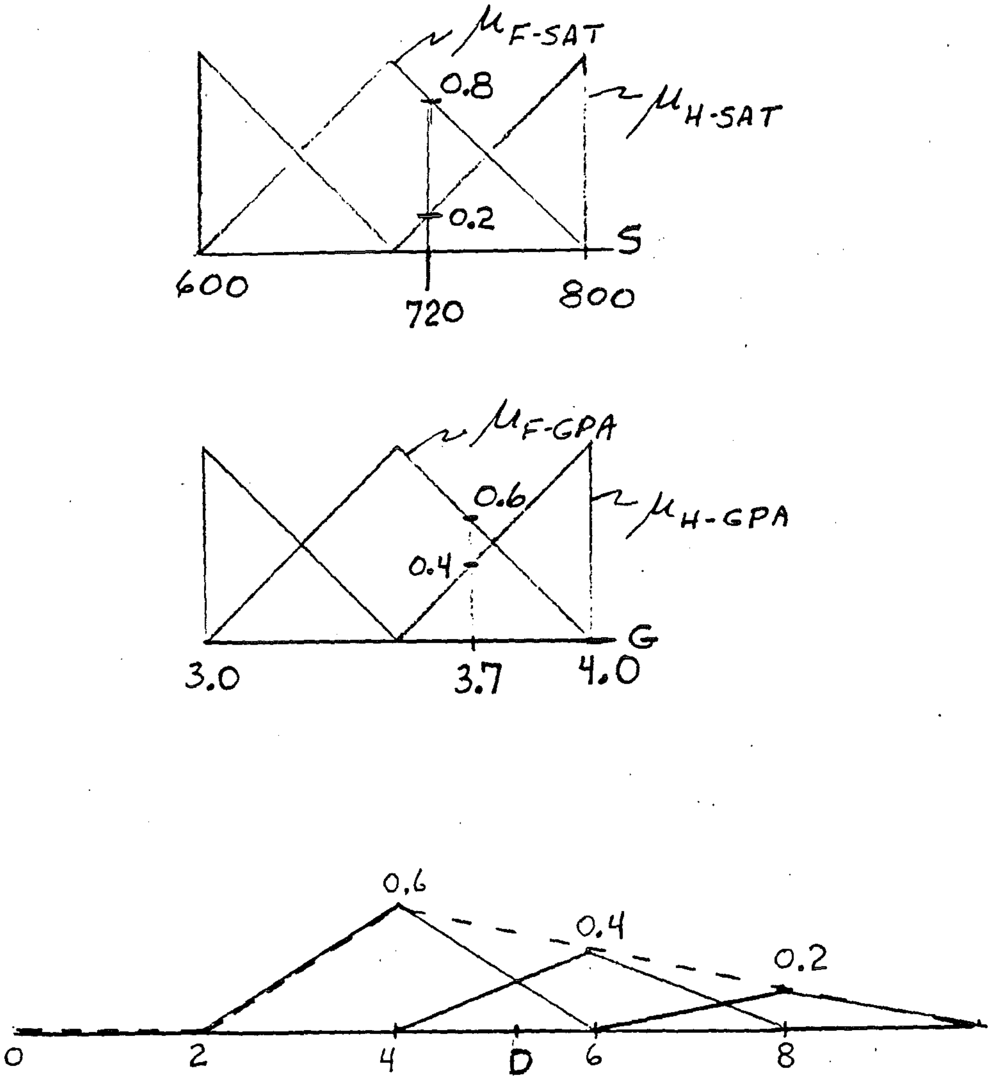

Figure 2: Student X has an GRE score of 720. Thus, as shown in the top figure, Student X's membership in the set of fair GPA's is 0.2 and high GRE scores is 0.5. Since Student X's GPA is 3.7, his/her membership in the set of high GPA's is 0.6 and in very high GPA's is 0.4. The weighted consequent membership functions, shown on the bottom, yield 0.0 for poor, 0.2 for average, 0.6 for good and 0.4 for excellent. The center of mass for the sum of these two weighted curves is the defuzzified consequent. The result is 5.33, roughly half way between average and good.

In summary, the the weights of the consequents for the running example are

$$\begin{array}{lcll} \mathrm{P} & \Rightarrow & \max(0,0) & = 0 \\ \mathrm{A} & \Rightarrow & \min(0.6,0.8) & = 0.6 \\ \mathrm{G} & \Rightarrow & \max[\,\min(0.6,0.2),\,\min(0.4,0.8] & = 0.4 \\ \mathrm{E} & \Rightarrow & \min(0.4,0.2) & = 0.2 \end{array}$$

1.2.2 Defuzzification: Finding the Crisp Consequent

The goal, however, is to have a single assessment of the performance forecast for Student X. The four numerical consequents can be combined into a single crisp assessment. The process of doing so is referred to as *defuzzification*.

With reference to the bottom membership functions in Figure 1, let $\mu_{\mathrm{P}}(x)$ be the membership function for P, $\mu_{\mathrm{A}}(x)$ be for A and $\mu_{\mathrm{G}}(x)$ be for G and $\mu_{\mathrm{E}}(x)$ for E. Defuzzification is performed by specifying a measure of central tendancy of the consequent membership functions when the P membership function is assigned a value of 0.0, the A membership function a value of 0.2, G a value of 0.6 and the E membership function a value of 0.4. There are a number of ways this can be done.

By a *measure of central tendancy*, we mean an assessment of the 'middle' of the weighted membership functions. Recall the probability density function. Commonly used measures of central tendancy for PDF's include the mean, mode and median. If $p(x)$ is the probability density function, then

$$\text{mean} = \int_{-\infty}^{\infty} x\, p(x) dx,$$

$$\text{mode} = \arg\max p(x)$$

and the median is the solution to the equation

$$\int_{-\infty}^{\text{median}} p(x) dx = \int_{\text{median}}^{\infty} p(x) dx = \frac{1}{2}.$$

In many PDF's, the mean, mode and median are equal.

Defuzzification depends on the measure of central tendancy used. Two methods can be illustrated with the plot on the bottom of Figure 2. Each membership function is multiplied by its corresponding weight.

1. If the mode is used, the defuzzification predicts that Student X will be an 'average' graduate student. When the mode is used for defuzzification, the crisp consequent can be expressed as a linguistic variable.
2. If the mean is used, the linguistic varibles of P, A, G and E must be quantified. As illustrated in the plot on the bottom of Figure 2, let P have a numerical value of 2, A a value of 4, G a value of 6 and assign 8 to E. The function for which the center of mass (mean) is to be computed is

   $$f(x) = 0.0\mu_{\mathrm{P}}(x) + 0.2\mu_{\mathrm{A}}(x) + 0.4\mu_{\mathrm{G}}(x) + 0.6\mu_{\mathrm{E}}(x)$$

   The center of mass for the defuzzification, D, is

   $$\begin{aligned} D &= \frac{\int_{-\infty}^{\infty} x\, f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} \\ &= 5.33 \end{aligned}$$

Matrix Description

Fuzzy If-Then rules, in many cases, can be expressed in matrix form. We repeat the If-Then rules of the running example.

If GPA is H And GRE is H, Then E
Or, If [GPA is F And GRE is H,
Or GPA is H And GRE is F], Then G
Or, If GPA is F And GRE is F, Then A
Or, If [GPA is P
OR GRE is P], Then P

Alternately, an exhaustive list of antecedants can be made and the corresponding consequent assigned. For this example,

If GPA is H And GRE is H, Then E
Or, If GPA is H And GRE is F, Then G
Or, If GPA is H And GRE is P, Then P
Or, If GPA is F And GRE is H, Then G
Or, If GPA is F And GRE is F, Then A
Or, If GPA is F And GRE is P, Then P
Or, If GPA is P And GRE is H, Then P
Or, If GPA is P And GRE is F, Then P
Or, If GPA is P And GRE is P, Then P

This list of rules can, in turn, be expressed concisely in a linguistic rule matrix.

$$\mathbf{R} \Rightarrow \begin{array}{|c||c|c|c|} \hline \text{GRE} \Downarrow / \text{GPA} \Rightarrow & \text{L} & \text{F} & \text{H} \\ \hline \text{L} & \text{P} & \text{P} & \text{P} \\ \text{F} & \text{P} & \text{A} & \text{G} \\ \text{H} & \text{P} & \text{G} & \text{E} \\ \hline \end{array} \qquad (1)$$

1.3 Generalization

The previous example can be generalized. Let o_n be the nth object of the antecedent. Assume o_n is calibrated into K_n fuzzy membership functions. For example, o_1 is a GPA calibrated into $K_1 = 3$ membership functions. Let $\{\ell_{nk} | 1 \leq k \leq K_n\}$ be linguistic descriptors of o_n with corresponding membership functions $\{\mu_{nk}(o_n) | 1 \leq k \leq K_n\}$. The fuzzy if-then rules can then be written as

$$\text{If } o_1 \text{ is } \vec{\ell}_1 \text{ and } o_2 \text{ is } \vec{\ell}_2 \text{ and } \cdots \text{ and } o_n \text{ is } \vec{\ell}_n \text{ and } \cdots \text{ and } o_N \text{ is } \vec{\ell}_N, \text{ then } c \text{ is } \mathbf{R}_{K_1,K_2,\cdots K_N}$$

where c is the consequent described by the relationship $\mathbf{R}$, N is the number of antecedents and $\vec{\ell}_n$ is a vector whose components are ℓ_{nk}. The $\mathbf{R}$ matrix for the running example is shown in Equation 1.

Let the J entries of $\mathbf{R}$ be calibrated using J membership functions, $\{\mu_j(x) | 1 \leq j \leq J\}$. Let the weight of the jth membership function have weight w_j. The center of mass defuzzification follows as

$$d = \frac{\int_{-\infty}^{\infty} x \sum_{j=1}^{J} w_j \mu_j(x) dx}{\int_{-\infty}^{\infty} \sum_{j=1}^{J} w_j \mu_j(x) dx} \qquad (2)$$

Let the area of the jth membership function be denoted by

$$A_j = \int_{-\infty}^{\infty} \mu_j(x)dx$$

The center of mass of the jth membership function is

$$m_j = \int_{-\infty}^{\infty} x\frac{\mu_j(x)}{A_j}dx$$

The defuzzification in Equation 2 can then be written as

$$d = \frac{\sum_{j=1}^{J} w_j A_j m_j}{\sum_{j=1}^{J} w_j A_j} \tag{3}$$

In many cases, all of the areas are equal. In such cases

$$d = \frac{\sum_{j=1}^{J} w_j m_j}{\sum_{j=1}^{J} w_j}$$

Consider, again, defuzzification of the weighted membership functions shown at the bottom for Figure 2. Clearly, the areas of all membership functions are equal. The center of massses are

$$m_P = 2, m_A = 4, m_G = 6, m_E = 8$$

Thus

$$\begin{aligned} d &= \frac{0 \times 2 + 4 \times 0.6 + 6 \times 0.4 + 8 \times 0.2}{0.6 + 0.4 + 0.2} \\ &= 5.33 \end{aligned} \tag{4}$$

or, roughly half way between average and good.

1.4 Variations

There exist numerous variations on the operations in fuzzy inferencing. Here are a few.

- Operations other than min and max can be used for the fuzzy inferencing. *Sum-product inferencing* uses multipication for the fuzzy and and addition for the fuzzy or.

- Defuzzification by clipping the membership function rather than weighting is commonly used. An example is shown in Figure 3 for the student assessment example in the previous section. (Compare to the defuzzification on the bottom of Figure 2.

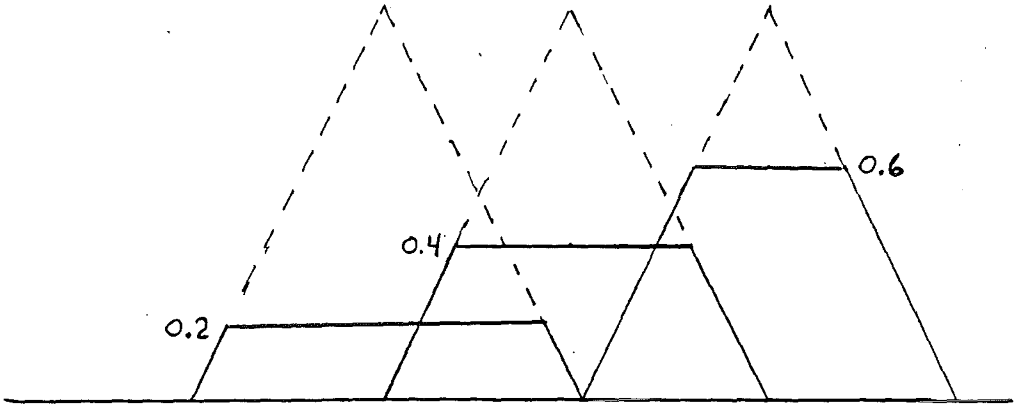

Figure 3: An alternate method of defuzzification. The membership functions are clipped and the corresponding center of mass evaluated.

Problems

1. Assess the forecasted performance of Student X using Yager logic with a value of w other that infinity. Use membership weighting and center of mass for defuzzification.
2. Assess Student X using sum-product inferencing. Comment on the need, if any, to recalibrate the consequent membership functions.
3. Assess Student X using the weighted membership functions in Figure 2 when the median is used for defuzzification.
4. Generate a defuzzification formula of the type in Equation 3 when, as illustrated in Figure 3, clipping is used. Assume defuzzification is performed using the center of mass.
5. What is the change in the assessment of Student X when clipping is used in center of mass defuzzification?
6. Consider fuzzification of a number followed immediately by defuzzification using the same set of fuzzy membership functions. If the membership functions are, say, Gaussian in shape, then the defuzzification will be different that the original number. What are conditions on the membership function shapes that will result in these values being equivalent?

Fuzzy Inference Engines

Robert J. Marks II

1 Inference Engines

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E	for	**excellent**
F	for	**fair**
G	for	**good**
H	for	**high,**
L	for	**low,**
P	for	**poor**

The If-Then Rules can then be written as

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$$\mu_{\text{F-GRE}}(720) = 0.8$$

$$\mu_{\text{H-GRE}}(720) = 0.2$$

If Student X has an undergraduate GPA of 3.7, the corresponding membership functions have values of

$$\mu_{\text{L-GPA}}(3.7) = 0$$

$$\mu_{\text{F-GPA}}(3.7) = 0.6,$$

$$\mu_{\text{H-GPA}}(3.7) = 0.4.$$

Assume the minimum operation if used for the fuzzy 'and'. (Other fuzzy intersetion and union operations can also be used.) Then the the composite membership for the antecedent, GPA is H And GRE is H, is

$$\begin{aligned}\mu_{\text{GPA is H And GRE is H}} &= \min[\mu_{\text{H-GPA}}(3.7), \mu_{\text{H-GRE}}(720) \\ &= \min(0.4, 0.2) \\ &= 0.2\end{aligned}$$

The consequent for a G graduate student is a bit longer

$$\begin{aligned}\mu_{\text{[GPA is F And GRE is H] Or [GPA is H And GRE is F]}} &= \max[\min(0.6, 0.2), \min(0.4, 0.8)] \\ &= 0.4\end{aligned}$$

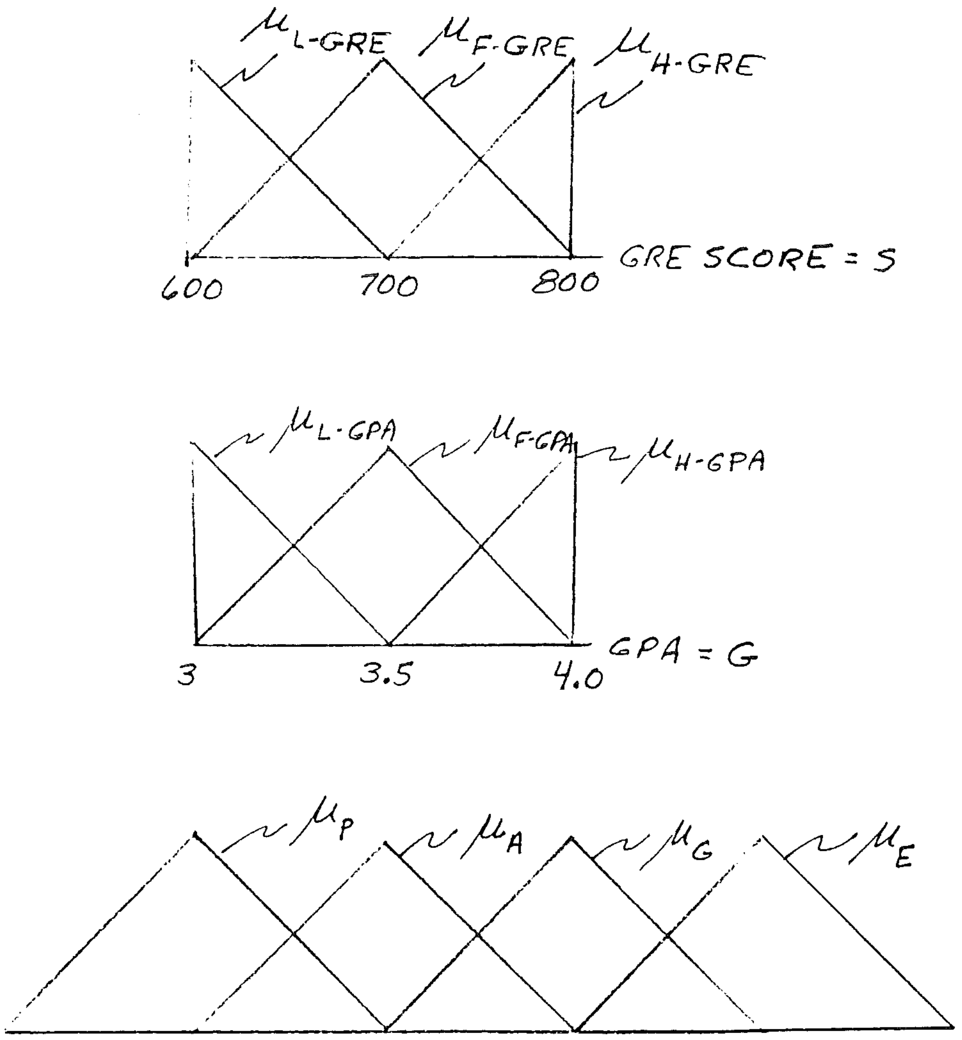


Figure 1: (top) Fuzzy membership functions for L, F and H scores for the analytic GRE test; (center) fuzzy membership functions for L, F and H GPA's; (bottom) fuzzy membership functions for P, A, G and E prospects for graduate student success.

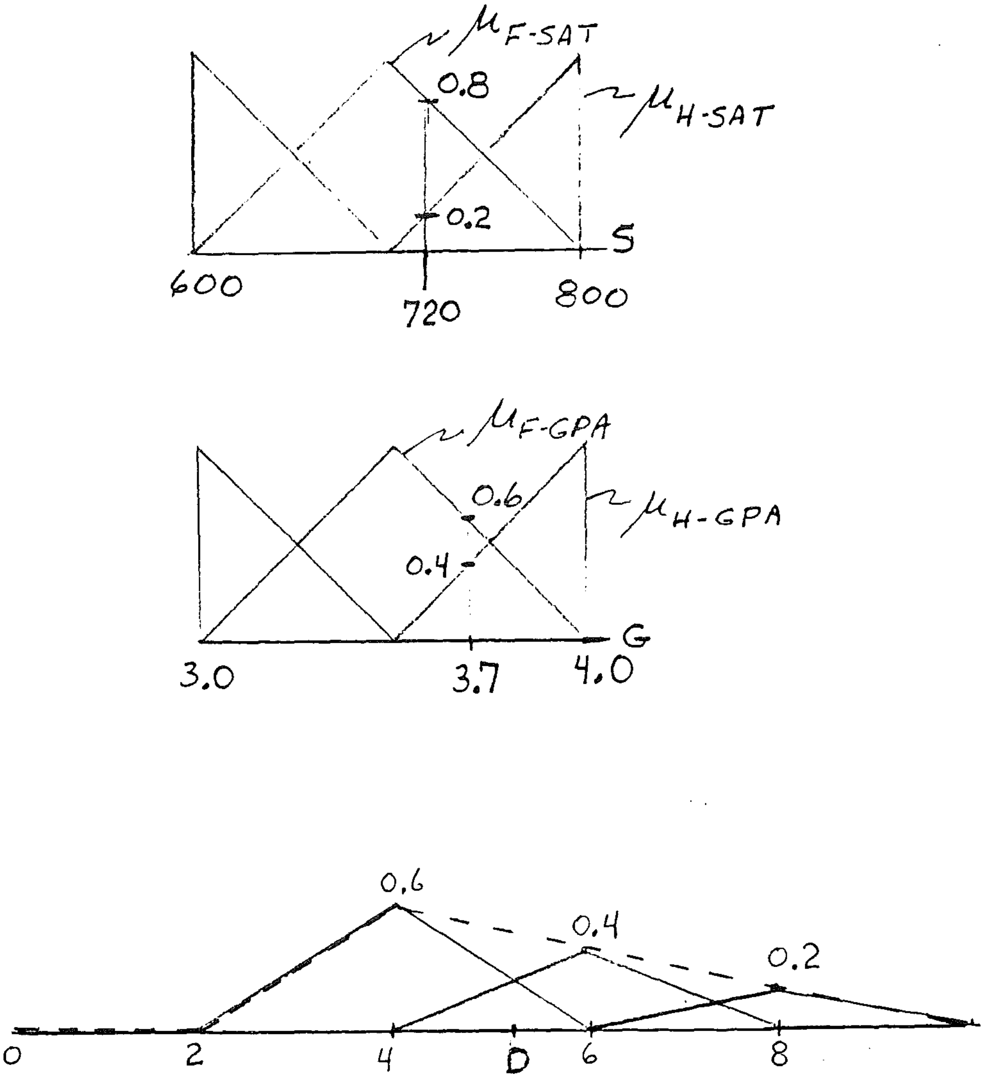


Figure 2: Student X has an GRE score of 720. Thus, as shown in the top figure, Student X's membership in the set of fair GPA's is 0.2 and high GRE scores is 0.5. Since Student X's GPA is 3.7, his/her membership in the set of high GPA's is 0.6 and in very high GPA's is 0.4. The weighted consequent membership functions, shown on the bottom, yield 0.0 for poor, 0.2 for average, 0.6 for good and 0.4 for excellent. The center of mass for the sum of these two weighted curves is the defuzzified consequent. The result is 5.33, roughly half way between average and good.

In summary, the the weights of the consequents for the running example are

$$\begin{array}{llll} \text{P} & \Rightarrow & \max(0,0) & = & 0 \\ \text{A} & \Rightarrow & \min(0.6,0.8) & = & 0.6 \\ \text{G} & \Rightarrow & \max[\ \min(0.6,0.2), \min(0.4,0.8] & = & 0.4 \\ \text{E} & \Rightarrow & \min(0.4,0.2) & = & 0.2 \end{array}$$

1.2.2 Defuzzification: Finding the Crisp Consequent

The goal, however, is to have a single assessment of the performance forecast for Student X. The four numerical consequents can be combined into a single crisp assessment. The process of doing so is referred to as *defuzzification*.

With reference to the bottom membership functions in Figure 1, let $\mu_{\text{P}}(x)$ be the membership function for P, $\mu_{\text{A}}(x)$ be for A and $\mu_{\text{G}}(x)$ be for G and $\mu_{\text{E}}(x)$ for E. Defuzzification is performed by specifying a measure of central tendancy of the consequent membership functions when the FtP membership function is assigned a value of 0.0, the A membership function a value of 0.2, G a value of 0.6 and the E membership function a value of 0.4. There are a number of ways this can be done.

By a *measure of central tendancy*, we mean an assessment of the 'middle' of the weighted membership functions. Recall the probability density function. Commonly used measures of central tendancy for PDF's include the mean, mode and median. If $p(x)$ is the probability density function, then

$$\text{mean} = \int_{-\infty}^{\infty} x\ p(x)dx,$$

$$\text{mode} = \arg\max p(x)$$

and the median is the solution to the equation

$$\int_{-\infty}^{\text{median}} p(x)dx = \int_{\text{median}}^{\infty} p(x)dx = \frac{1}{2}.$$

In many PDF's, the mean, mode and median are equal.

Defuzzification depends on the measure of central tendancy used. Two methods can be illustrated with the plot on the bottom of Figure 2. Each membership function is multiplied by its corresponding weight.

1. If the mode is used, the defuzzification predicts that Student X will be an 'average' graduate student. When the mode is used for defuzzification, the crisp consequent can be expressed as a linguistic variable.

2. If the mean is used, the linguistic varibles of P, A, G and E must be quantified. As illustrated in the plot on the bottom of Figure 2, let P have a numerical value of 2, A a value of 4, G a value of 6 and assign 8 to E. The function for which the center of mass (mean) is to be computed is

$$f(x) = 0.0\mu_{\text{P}}(x) + 0.2\mu_{\text{A}}(x) + 0.4\mu_{\text{G}}(x) + 0.6\mu_{\text{E}}(x)$$

The center of mass for the defuzzification, D, is

$$\begin{aligned} D &= \frac{\int_{-\infty}^{\infty} x\ f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} \\ &= 5.33 \end{aligned}$$

Matrix Description

Fuzzy If-Then rules, in many cases, can be expressed in matrix form. We repeat the If-Then rules of the running example.

If GPA is H And GRE is H, Then E
Or, If [GPA is F And GRE is H,
Or GPA is H And GRE is F], Then G
Or, If GPA is F And GRE is F, Then A
Or, If [GPA is P
OR GRE is P], Then P

Alternately, an exhaustive list of antecedants can be made and the corresponding consequent assigned. For this example,

If GPA is H And GRE is H, Then E
Or, If GPA is H And GRE is F, Then G
Or, If GPA is H And GRE is P, Then P
Or, If GPA is F And GRE is H, Then G
Or, If GPA is F And GRE is F, Then A
Or, If GPA is F And GRE is P, Then P
Or, If GPA is P And GRE is H, Then P
Or, If GPA is P And GRE is F, Then P
Or, If GPA is P And GRE is P, Then P

This list of rules can, in turn, be expressed concisely in a linguistic rule matrix.

$$\mathbf{R} \Rightarrow \begin{array}{|c||c|c|c|} \hline \text{GRE} \Downarrow / \text{GPA} \Rightarrow & \text{L} & \text{F} & \text{H} \\ \hline \text{L} & \text{P} & \text{P} & \text{P} \\ \text{F} & \text{P} & \text{A} & \text{G} \\ \text{H} & \text{P} & \text{G} & \text{E} \\ \hline \end{array} \qquad (1)$$

1.3 Generalization

The previous example can be generalized. Let o_n be the nth object of the antecedent. Assume o_n is calibrated into K_n fuzzy membership functions. For example, o_1 is a GPA calibrated into $K_1 = 3$ membership functions. Let $\{\ell_{nk} | 1 \leq k \leq K_n\}$ be linguistic descriptors of o_n with corresponding membership functions $\{\mu_{nk}(o_n) | 1 \leq k \leq K_n\}$. The fuzzy if-then rules can then be written as

$$\text{If } o_1 \text{ is } \vec{\ell}_1 \text{ and } o_2 \text{ is } \vec{\ell}_2 \text{ and } \cdots \text{ and } o_n \text{ is } \vec{\ell}_n \text{ and } \cdots \text{ and } o_N \text{ is } \vec{\ell}_N, \text{ then } c \text{ is } \mathbf{R}_{K_1, K_2, \cdots K_N}$$

where c is the consequent described by the relationship $\mathbf{R}$, N is the number of antecedents and $\vec{\ell}_n$ is a vector whose components are ℓ_{nk}. The $\mathbf{R}$ matrix for the running example is shown in Equation 1.

Let the J entries of $\mathbf{R}$ be calibrated using J membership functions, $\{\mu_j(x) | 1 \leq j \leq J\}$. Let the weight of the jth membership function have weight w_j. The center of mass defuzzification follows as

$$d = \frac{\int_{-\infty}^{\infty} x \sum_{j=1}^{J} w_j \mu_j(x) dx}{\int_{-\infty}^{\infty} \sum_{j=1}^{J} w_j \mu_j(x) dx} \qquad (2)$$

Let the area of the jth membership function be denoted by

$$A_j = \int_{-\infty}^{\infty} \mu_j(x) dx$$

The center of mass of the jth membership function is

$$m_j = \int_{-\infty}^{\infty} x \frac{\mu_j(x)}{A_j} dx$$

The defuzzification in Equation 2 can then be written as

$$d = \frac{\sum_{j=1}^{J} w_j A_j m_j}{\sum_{j=1}^{J} w_j A_j} \tag{3}$$

In many cases, all of the areas are equal. In such cases

$$d = \frac{\sum_{j=1}^{J} w_j m_j}{\sum_{j=1}^{J} w_j}$$

Consider, again, defuzzification of the weighted membership functions shown at the bottom for Figure 2. Clearly, the areas of all membership functions are equal. The center of massses are

$$m_P = 2, m_A = 4, m_G = 6, m_E = 8$$

Thus

$$\begin{aligned} d &= \frac{0 \times 2 + 4 \times 0.6 + 6 \times 0.4 + 8 \times 0.2}{0.6 + 0.4 + 0.2} \\ &= 5.33 \end{aligned} \tag{4}$$

or, roughly half way between average and good.

1.4 Variations

There exist numerous variations on the operations in fuzzy inferencing. Here are a few.

- Operations other than min and max can be used for the fuzzy inferencing. *Sum-product inferencing* uses multipication for the fuzzy and and addition for the fuzzy or.
- Defuzzification by clipping the membership function rather than weighting is commonly used. An example is shown in Figure 3 for the student assessment example in the previous section. (Compare to the defuzzification on the bottom of Figure 2.

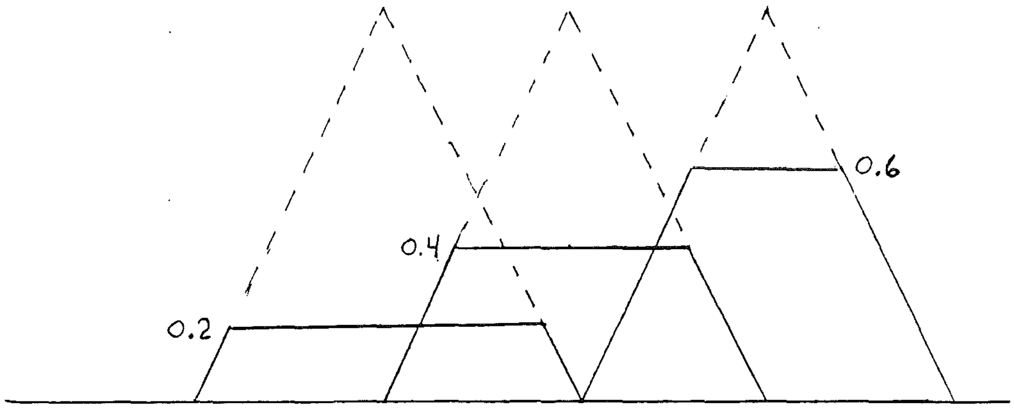
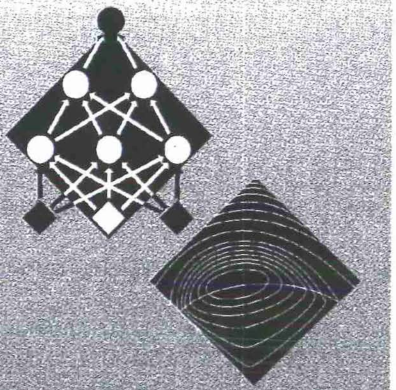


Figure 3: An alternate method of defuzzification. The membership functions are clipped and the corresponding center of mass evaluated.

Problems

1. Assess the forecasted performance of Student X using Yager logic with a value of w other that infinity. Use membership weighting and center of mass for defuzzification.
2. Assess Student X using sum-product inferencing. Comment on the need, if any, to recalibrate the consequent membership functions.
3. Assess Student X using the weighted membership functions in Figure 2 when the median is used for defuzzification.
4. Generate a defuzzification formula of the type in Equation 3 when, as illustrated in Figure 3, clipping is used. Assume defuzzification is performed using the center of mass.
5. What is the change in the assessment of Student X when clipping is used in center of mass defuzzification?
6. Consider fuzzification of a number followed immediately by defuzzification using the same set of fuzzy membership functions. If the membership functions are, say, Gaussian in shape, then the defuzzification will be different that the original number. What are conditions on the membership function shapes that will result in these values being equivalent?



Fuzzy Parameter Adaptation in Optimization:

Some Neural Net Training Examples

Payman Arabshahi, *University of Alabama in Huntsville*
Jai J. Choi, *Boeing Computer Services*
Robert J. Marks II, *University of Washington*
Thomas P. Caudell, *University of New Mexico*

Parameters of certain neural net training algorithms and classification procedures are often chosen [illegible]

MANY NONLINEAR OPTIMIZATION ALGORITHMS, INCLUDING THOSE used to train various types of artificial neural networks, strive to optimize some performance measure through judicious selection of one or more parameters. For instance, in the backpropagation-trained multilayer perceptron,[1] the performance measure is convergence speed. This speed is affected by the choice of learning and momentum parameters. Similarly, in the Adaptive Resonance Theory (ART 1) network,[2] the choice of a vigilance parameter affects the number of classes into which the data are classified. The values of these parameters can be adapted during training to improve the performance measure(s) of the neural network. Table 1 summarizes the performance measures and parameters associated with several neural net architectures. The theme introduction on pp. 36–42 provides some background on these various methods.

Training parameters are typically chosen and adapted by a "neural smith," using human judgment, experience, and heuristic rules. For example, a smooth error surface in the backpropagation training of a layered perceptron suggests use of a long step, whereas a steep surface suggests smaller steps. Note that this description is fuzzy: the terms "smooth," "long," "steep," and "smaller" are each fuzzy linguistic variables.

Rather than choosing and optimizing these parameters manually, however, we take advantage of the fact that the linguistic variables used in human judgment can in many cases be quantified into a rule-based fuzzy inference engine. This fuzzy controller then replaces the neural smith. This methodology for choosing training parameters can be applied to other neural networks, including Kohonen's self-organizing maps[3] and layered perceptrons trained by other methods, such as random search.[4] But beyond neural nets, this research has led us to adopt the principles of fuzzy logic in a way that can potentially be broadly applied to a wide variety of algorithms used in adaptation and optimization.

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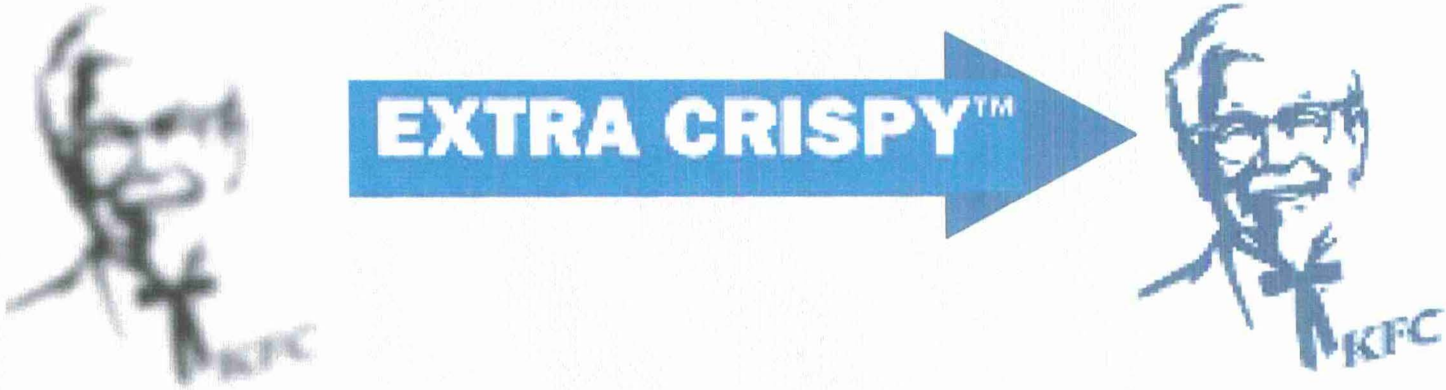
Fuzzy Logic

D. Fuzzy Set Properties and Definitions

Alpha Cuts

- α-cuts are used to make a fuzzy set crisp (8 9)



- α & β-cuts can be used for tri-valent logic.

Fuzzy Convex Combinations

- Crisp Convex Combinations 10
- Fuzzy Convex Combination of two fuzzy sets 11
- Properties
- Existence 12

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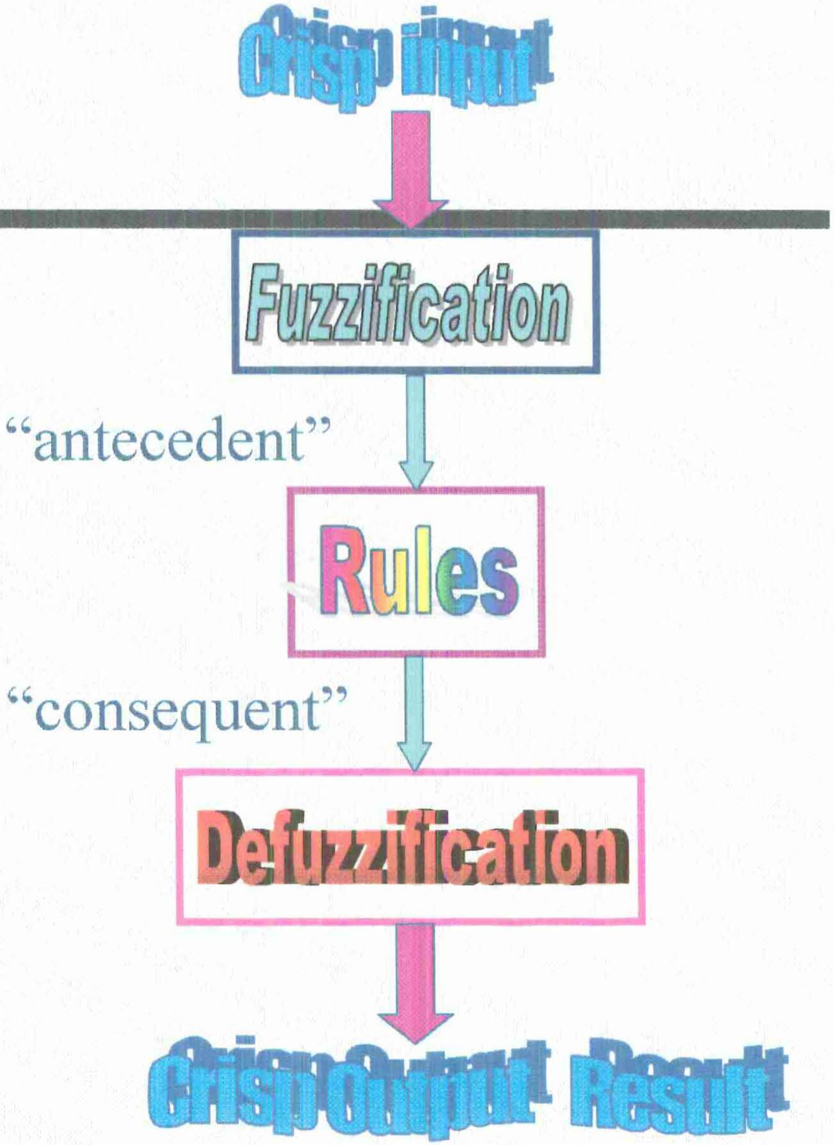
Fuzzy Logic

E. Fuzzy Inference Engine

Fuzzy Inference



Crisp input

Fuzzification

"antecedent"

Rules

"consequent"

Defuzzification

Crisp Output Result

Fuzzy Inference Example

- Assume that we need to evaluate student applicants based on their GPA and GRE scores.
- For simplicity, let us have three categories for each score [High (H), Medium (M), and Low(L)]
- Let us assume that the decision should be Excellent (E), Very Good (VG), Good (G), Fair (F) or Poor (P)
- An expert will associate the decisions to the GPA and GRE score. They are then Tabulated.

Fuzzy Inference Example

- **Fuzzy if-then Rules**

If the GRE is HIGH and the GPA is HIGH
then the student will be EXCELLENT.

If the GRE is LOW and the GPA is HIGH
then the student will be FAIR.

etc

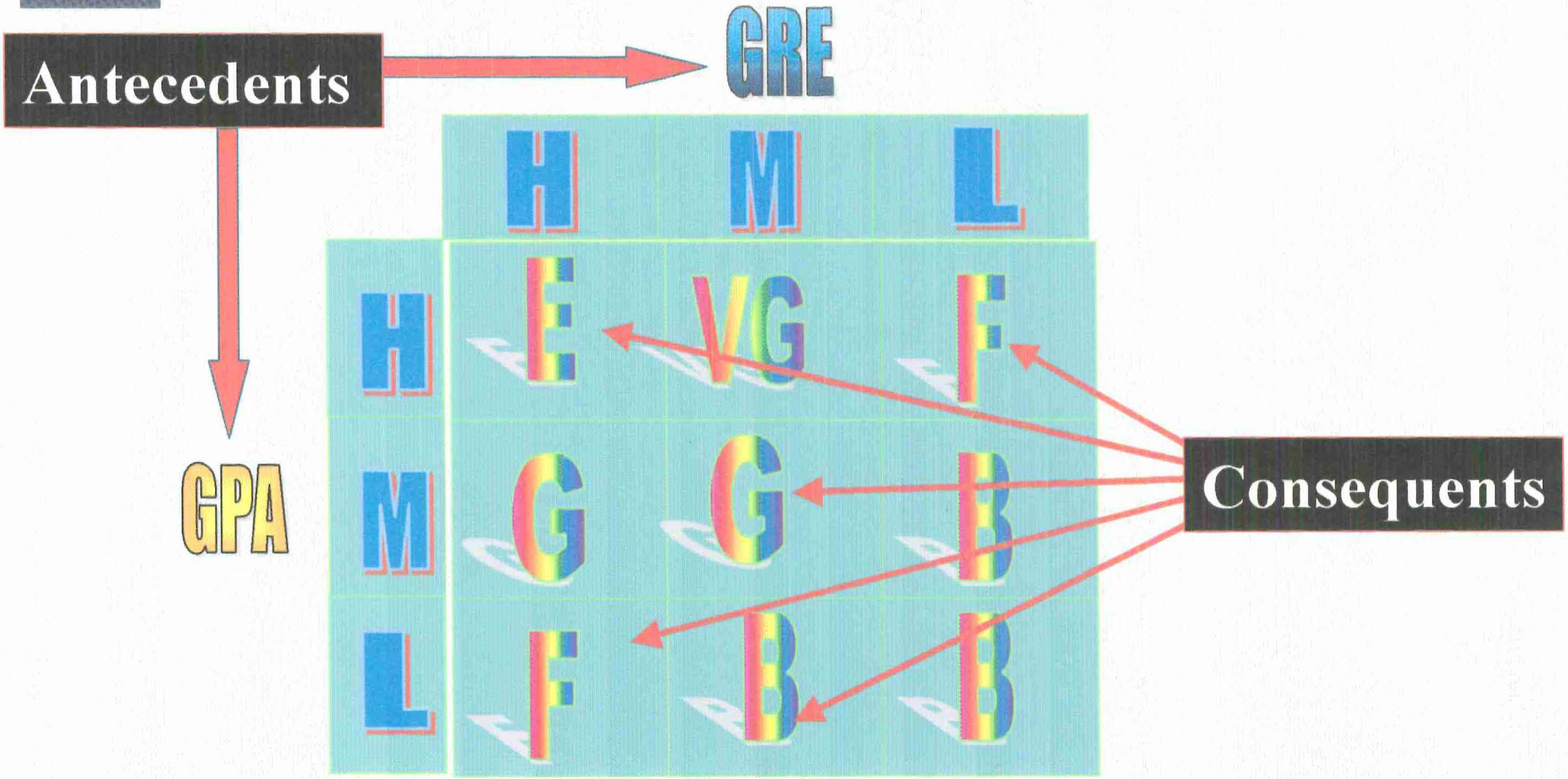
Fuzzy Linguistic Variables

Fuzzy Logic

Antecedent

Consequent

Fuzzy Rule Table



Antecedents

GRE

GPA

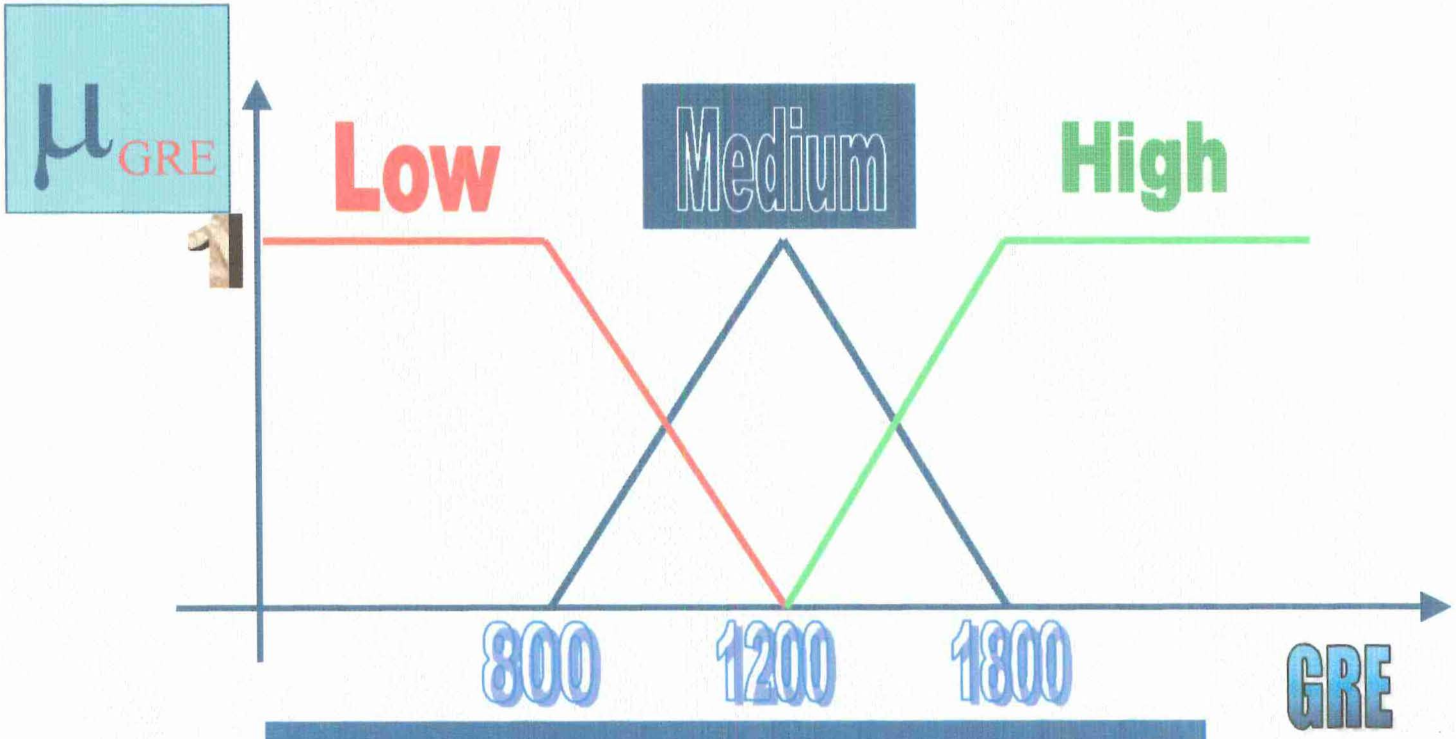
	H	M	L
H	E	VG	F
M	G	G	B
L	F	B	B

Consequents

Fuzzification

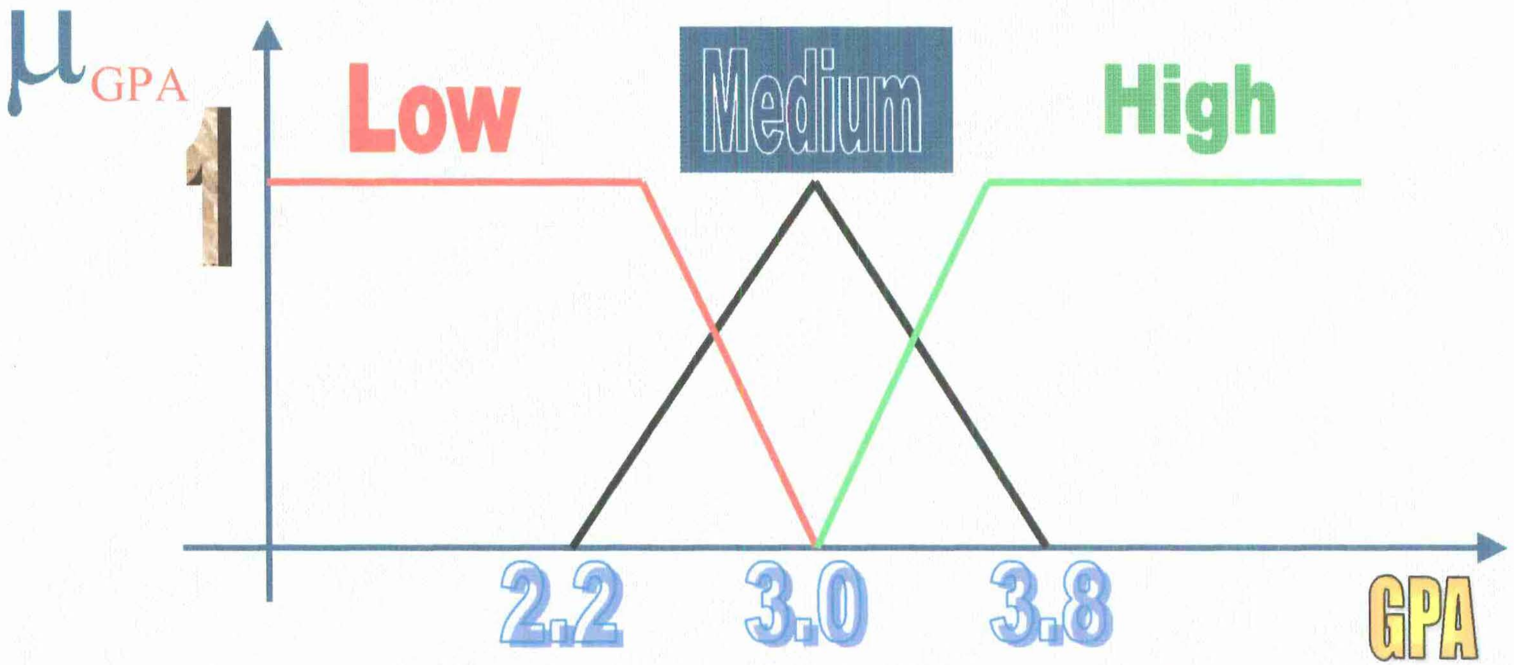
- **Fuzzifier converts a crisp input into a vector of fuzzy membership values.**
- **The membership functions**
    - reflects the designer's knowledge
    - provides smooth transition between fuzzy sets
    - are simple to calculate
- **Typical shapes of the membership function are Gaussian, trapezoidal and triangular.**

Membership Functions for GRE



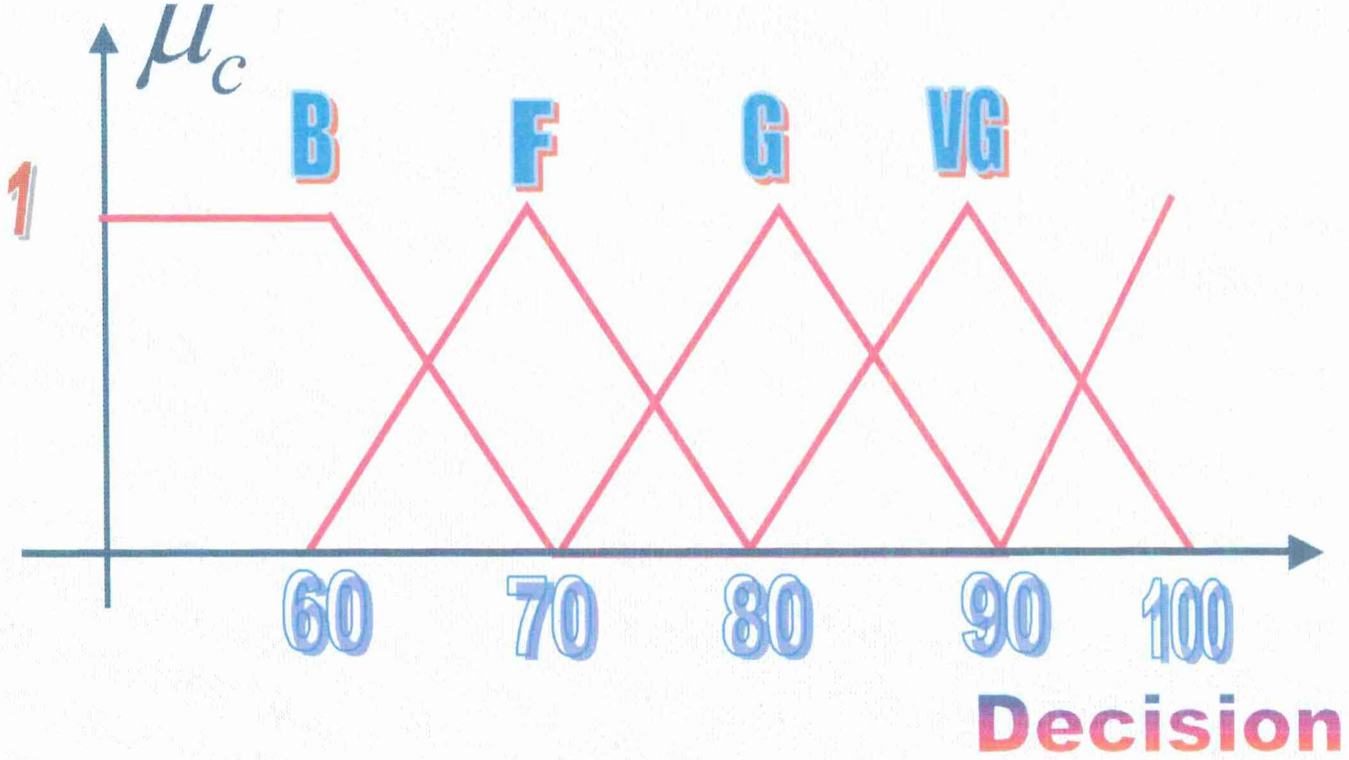
$$\mu_{GRE} = \{\mu_L, \mu_M, \mu_H\}$$

Membership Functions for the GPA



$$\mu_{GPA} = \{\mu_L, \mu_M, \mu_H\}$$

Membership Function for the Consequent



Fuzzification

- *Transform the crisp antecedents into a vector of fuzzy membership values.*
- *Assume a student with GRE=900 and GPA=3.6. Examining the membership function gives*

$$\mu_{\mathrm{GRE}} = \{\mu_{\mathrm{L}} = 0.8\,,\ \mu_{\mathrm{M}} = 0.2\,,\ \mu_{\mathrm{H}} = 0\}$$

$$\mu_{\mathrm{GPA}} = \{\mu_{\mathrm{L}} = 0\,,\ \mu_{\mathrm{M}} = 0.6\,,\ \mu_{\mathrm{H}} = 0.4\}$$

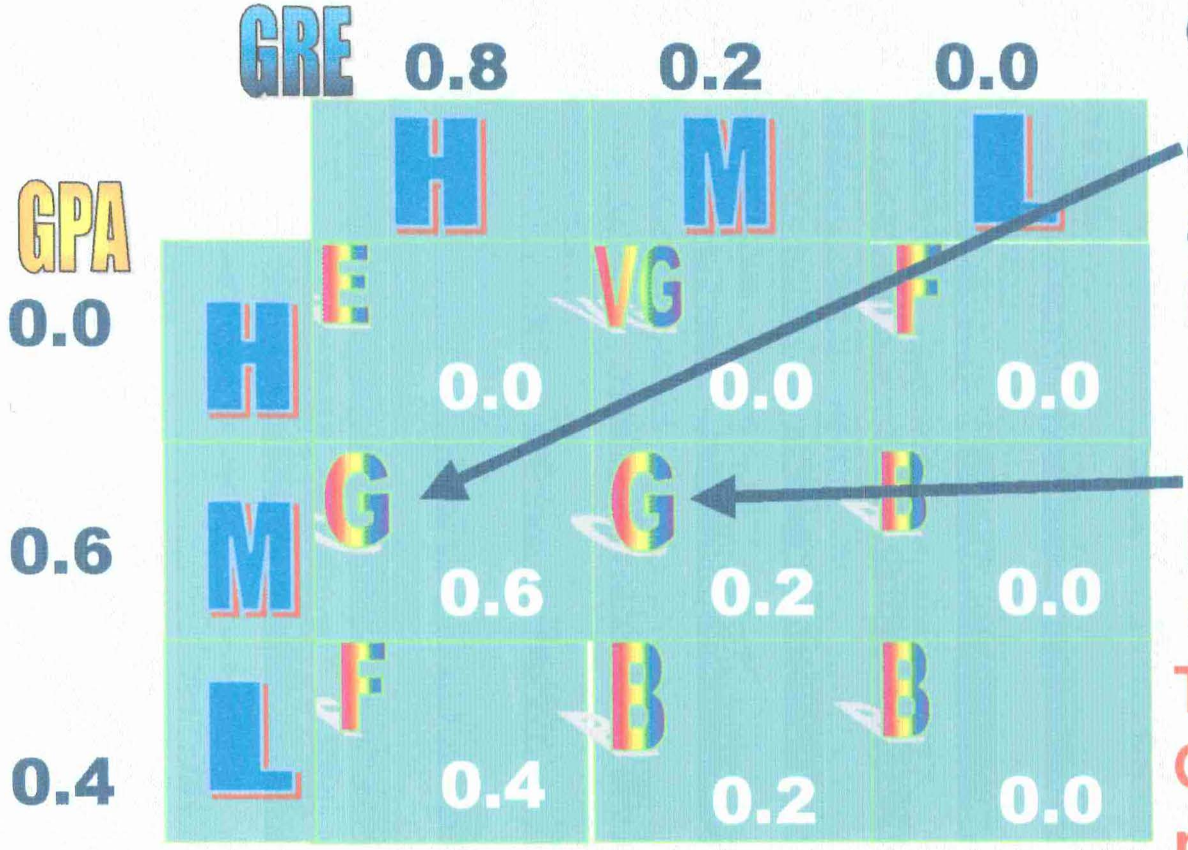
Table:

			GRE		
			0.8	0.2	0.0
			H	M	L
GPA	0.0	H	E	VG	F
	0.6	M	G	G	B
	0.4	L	F	B	B

Table:

				GRE	
			0.8	0.2	0.0
			H	M	L
GPA	0.0	H	E 0.0	VG 0.0	F 0.0
	0.6	M	G 0.6	G 0.2	B 0.0
	0.4	L	F 0.4	B 0.2	B 0.0

Interpretation:



	GRE	0.8	0.2	0.0
GPA		H	M	L
0.0	H	E 0.0	VG 0.0	F 0.0
0.6	M	G 0.6	G 0.2	B 0.0
0.4	L	F 0.4	B 0.2	B 0.0

The student is GOOD if

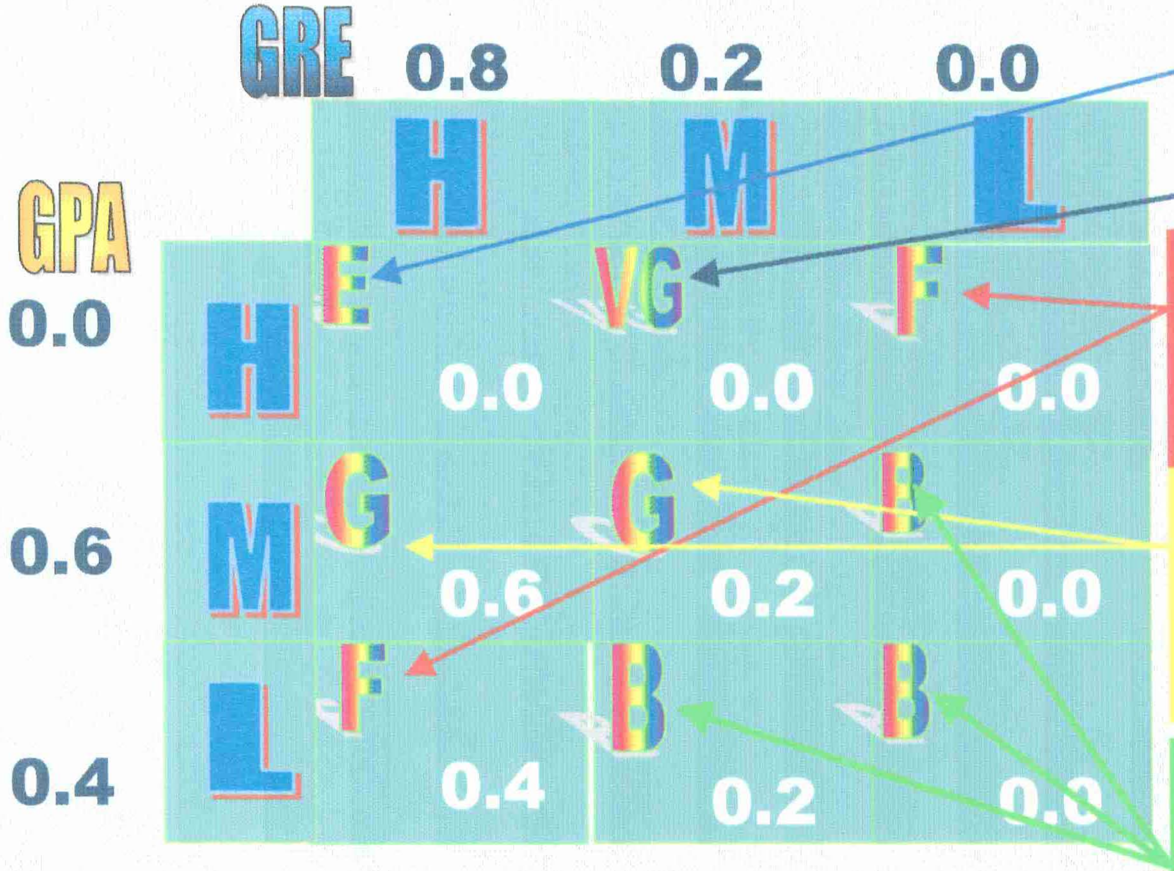
(the GRE is HIGH and the GPA is MEDIUM)

OR

(the GRE is MEDIUM and the GPA is MEDIUM)

The consequent GOOD has a membership of max(0.6,0.2)=0.6

Interpretation:



GPA \ GRE		0.8	0.2	0.0
		H	M	L
0.0	H	E 0.0	VG 0.0	F 0.0
0.6	M	G 0.6	G 0.2	B 0.0
0.4	L	F 0.4	B 0.2	B 0.0

$\mu_E = 0.0$

$\mu_{VG} = 0.0$

$\mu_F = \max(0.0, 0.4)$

$= 0.4$

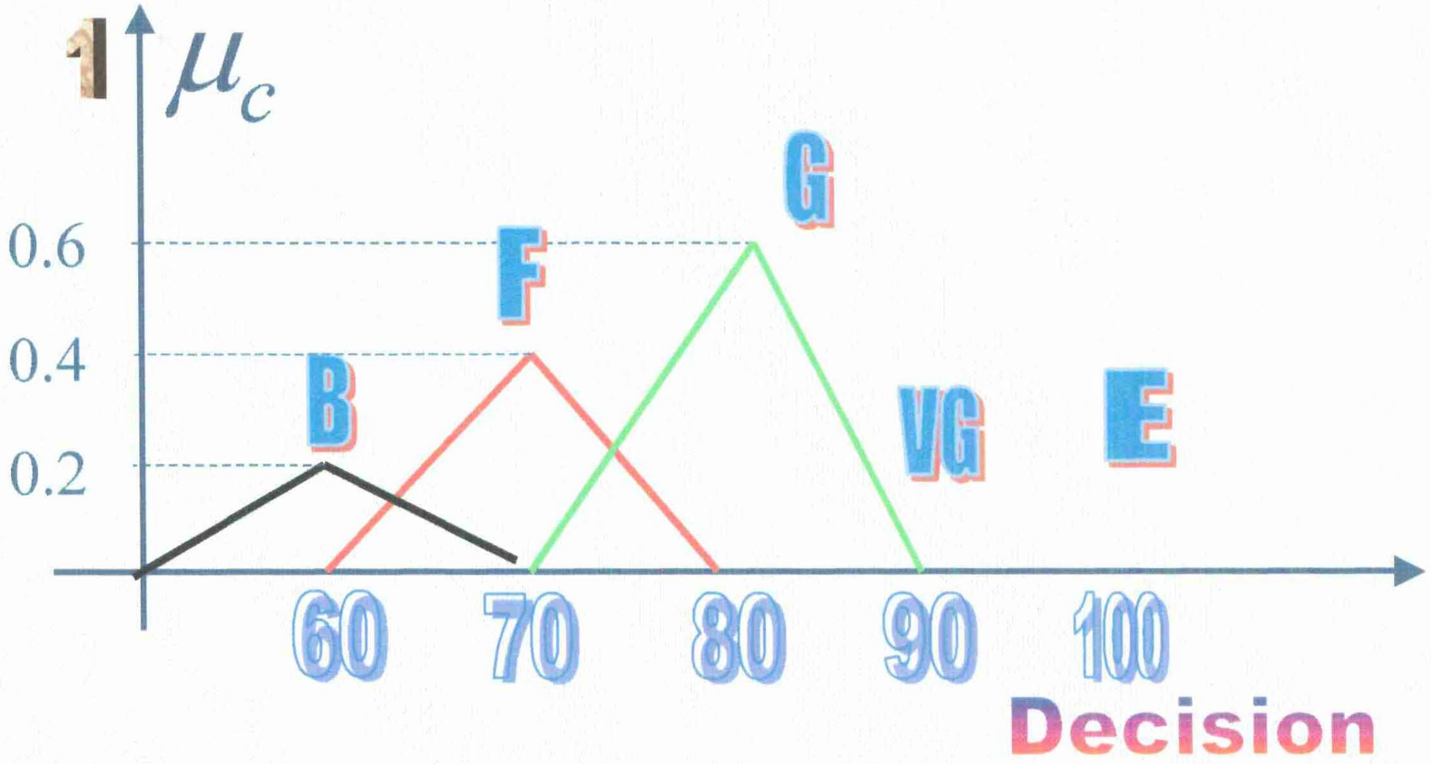
$\mu_G = \max(0.6, 0.2)$

$= 0.6$

$\mu_B = \max(0,0,0.2)$

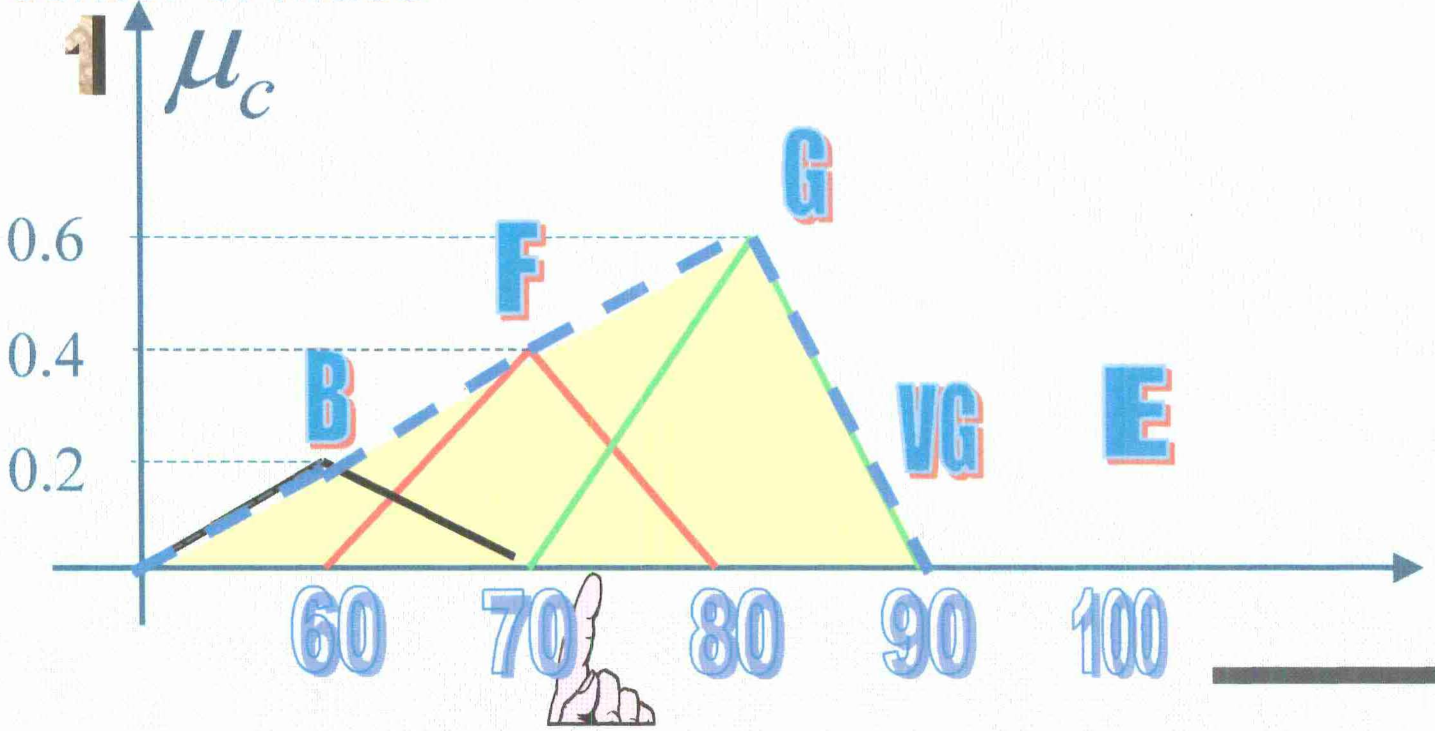
$= 0.2$

Weight Consequent Memberships



Defuzzification

- Converts the output fuzzy numbers into a unique (crisp) number
- Center of Mass Method: Add all weighted curves and find the center of mass



Mode Method

- An Alternate Approach: Fuzzy set with the largest membership value is selected.
- Fuzzy decision:

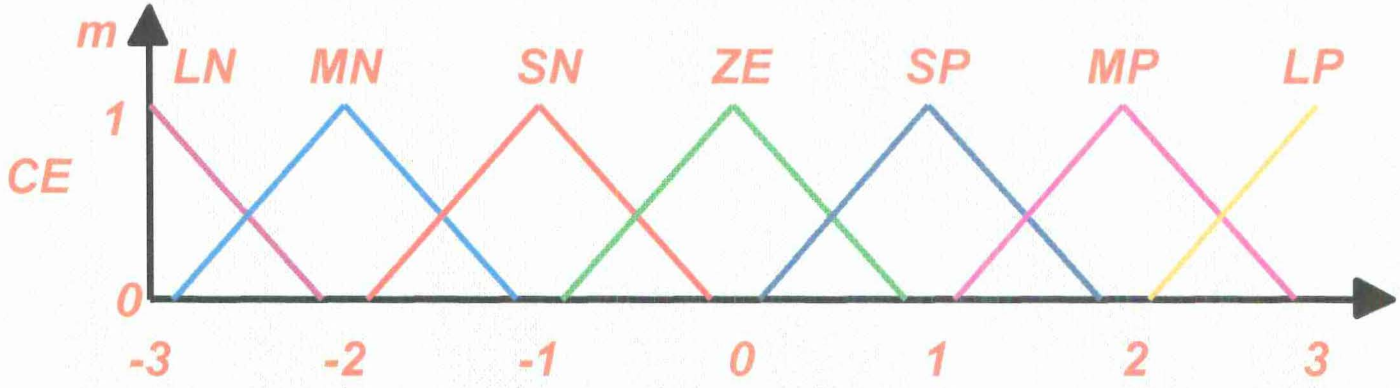
{B, F, G,VG, E} = {0.2, 0.4, 0.6, 0.0, 0.0}

- Final Decision (FD) = Fair Student
- If two decisions have same membership max, use the average of the two.

Example: Fuzzy Table for Control

					CE			
		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	MN	SN	SN
	MN	LN	LN	LN	MN	SN	ZE	ZE
	SN	LN	LN	MN	SN	ZE	ZE	SP
E	ZE	LN	MN	SN	ZE	SP	MP	LP
	SP	SN	ZE	ZE	SP	MP	LP	LP
	MP	ZE	ZE	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Membership Functions



Rule Aggregation

					CE			
		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	MN	e. SN 0.2	f. SN 0.0
	MN	LN	LN	LN	MN	d. SN 0.5	ZE	ZE
	SN	LN	LN	MN	c.SN 0.3	ZE	ZE	SP
E	ZE	LN	MN	b.SN 0.4	ZE	SP	MP	LP
	SP	a. SN 0.1	ZE	ZE	SP	MP	LP	LP
	MP	ZE	SP	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Consequent is or SN if *a* or *b* or *c* or *d* or *f*.

Rule Aggregation

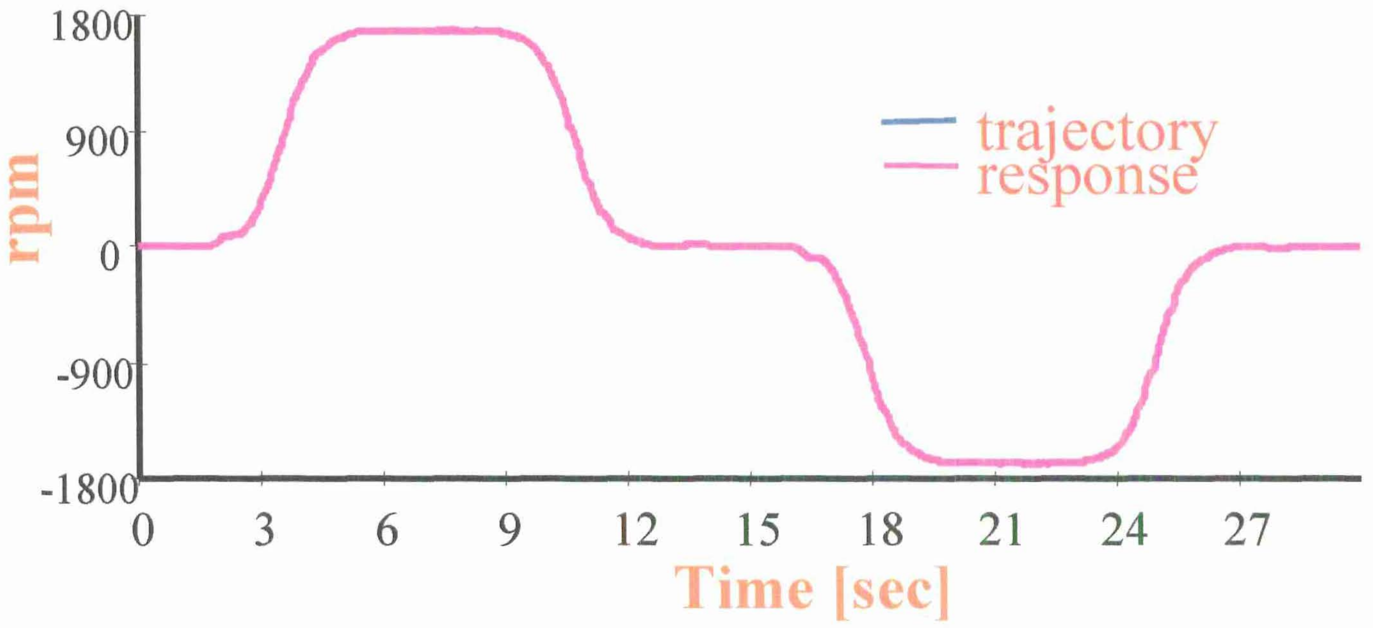
Consequent is or SN if *a* or *b* or *c* or *d* or *f.*

Consequent Membership = max(a,b,c,d,e,f) = 0.5

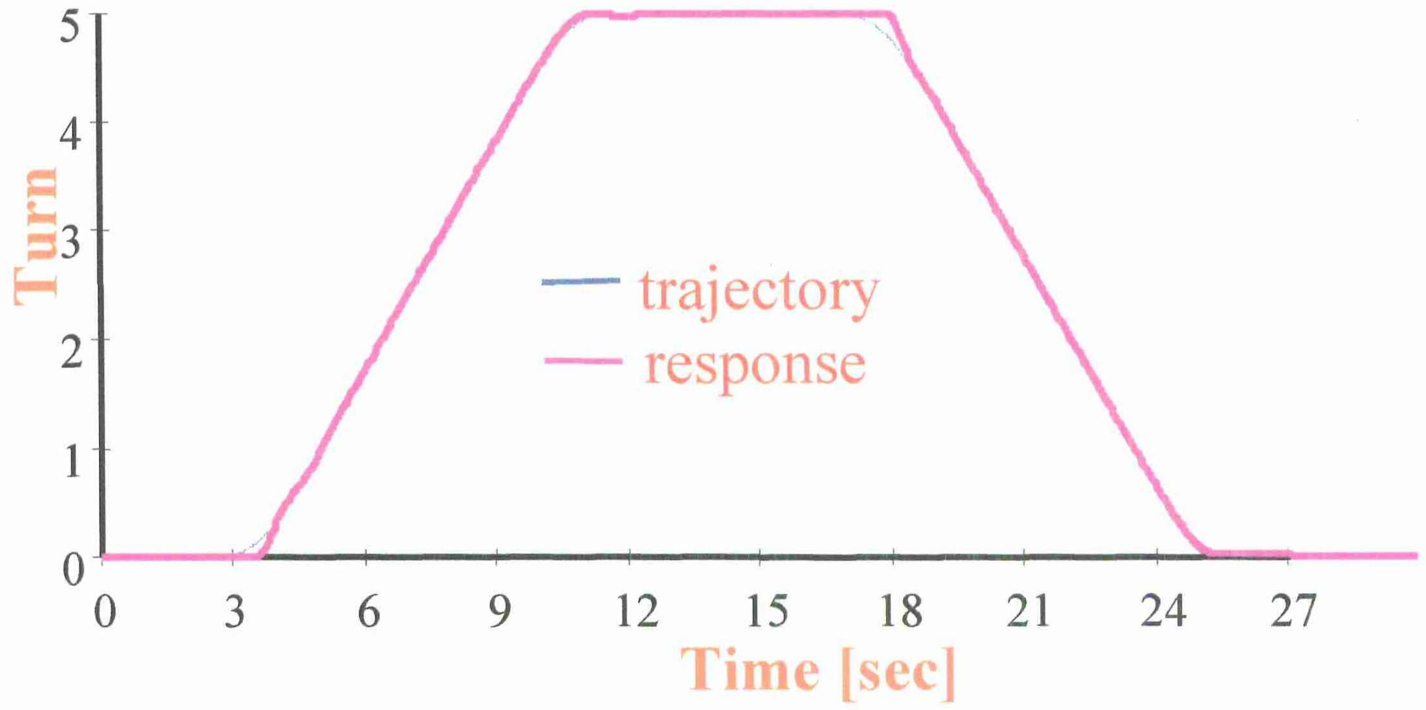
Use General Mean Aggregation:

$$agg_{\alpha}(\vec{x}) = \left[\frac{1}{N} \sum_{n=1}^{N} x_{n}^{\alpha} \right]^{1/\alpha}$$

Lab Test: Speed Tracking of IM

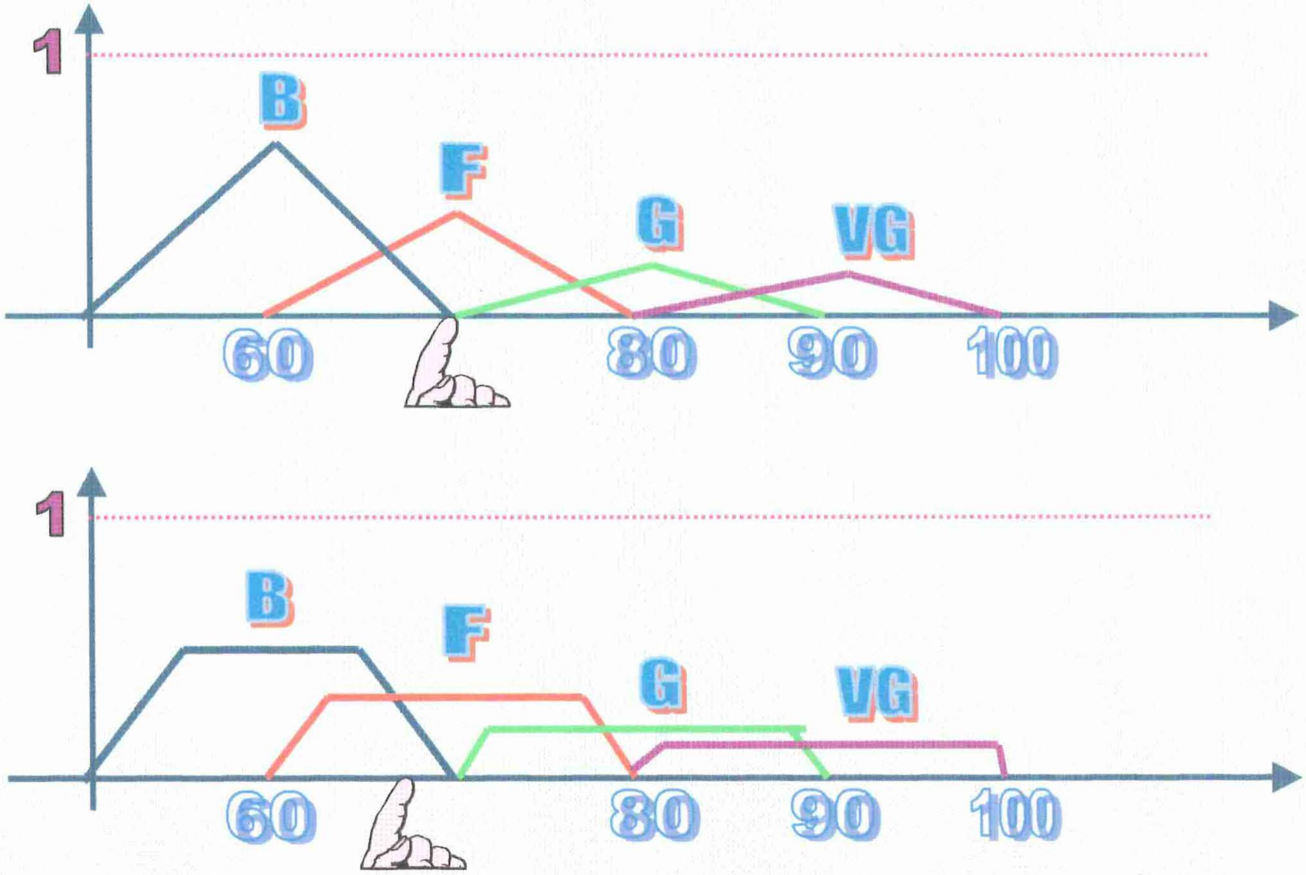


Lab Test: Prercision Position Tracking of IM



Commonly Used Variations

Clipped vs. Weighted Defuzzification



Commonly Used Variations

Sum-Product Inferencing

Instead of $\min(x,y)$ for fuzzy AND...

Use $\Rightarrow$ $x \bullet y$

Instead of $\max(x,y)$ for fuzzy OR...

Use $\Rightarrow$ $\min(1, x + y)$

Commonly Used Variations

Sugeno inferencing

Other Norms and co-norms

Relationship with Neural Networks

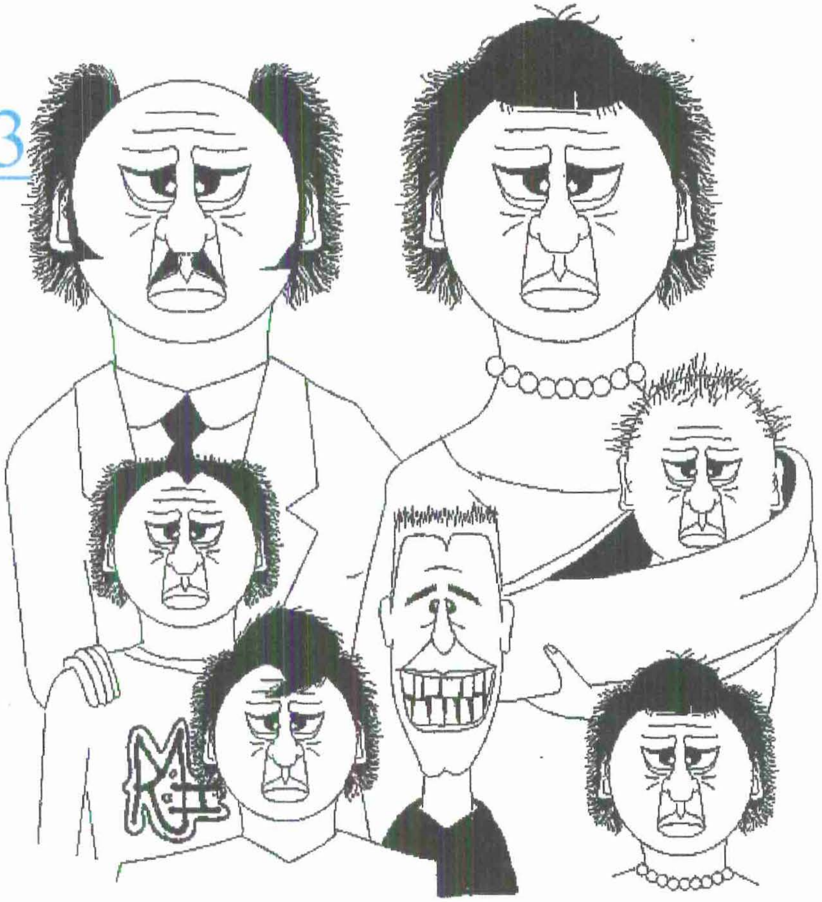
Explanation Facilities

Teaching a Fuzzy System

Tuning a Fuzzy System

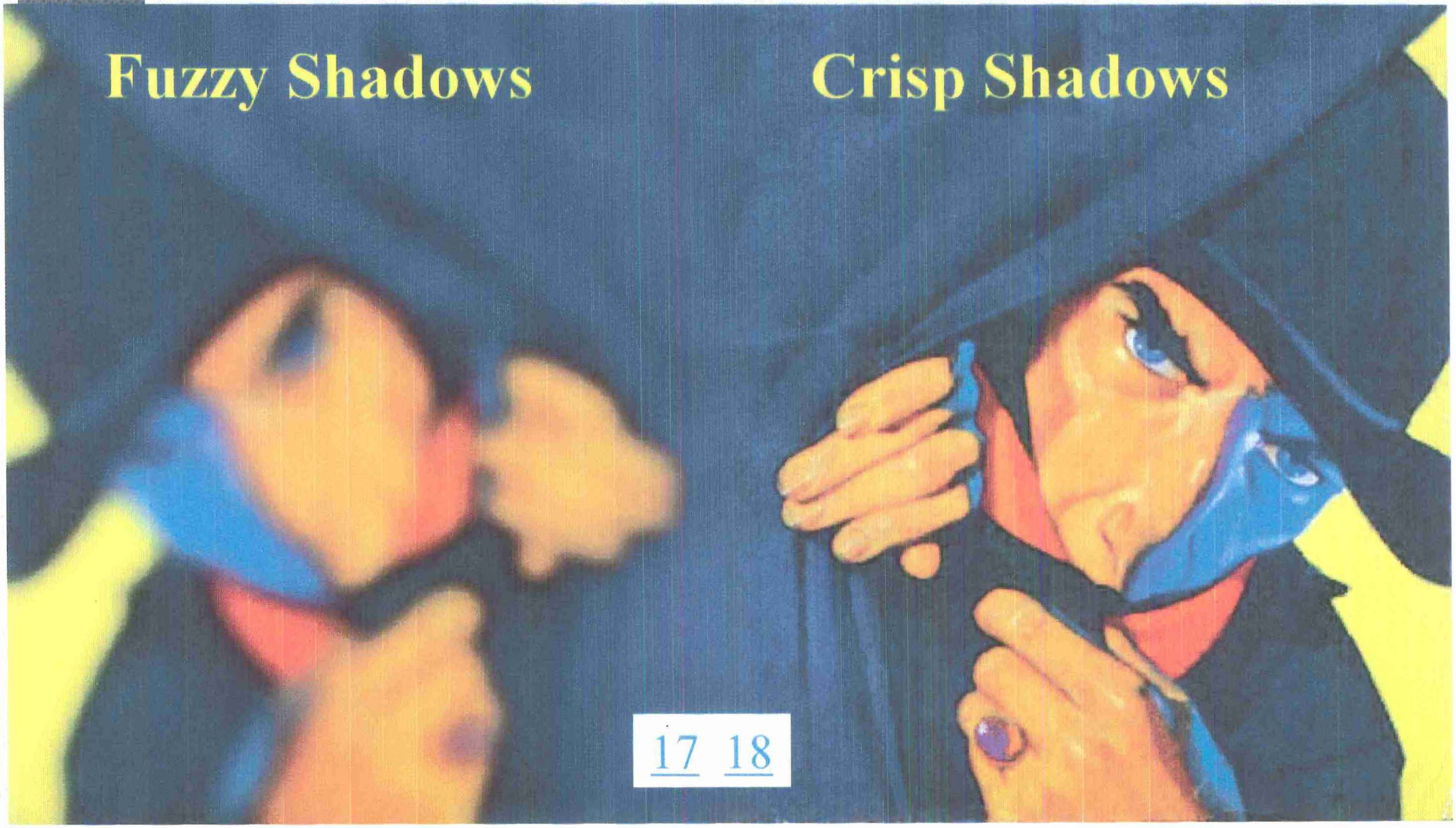
Fuzzy Relations 27

- Definition 19
- Composition 20 21 22 23
- Properties 24
- Sagittal Relations 25
- Sagittal Composition 26



Fuzzy Shadows:

Joint and Marginal Fuzzy Membership Functions



17 18

Fuzzy Convex Sets

Fuzzy Control Systems: Clear Advantages

Michael Reinfrank

Most of the concepts we use in daily life, such as large, small, heavy and light, are vague or "fuzzy." But machines must normally be provided with precise definitions. Fuzzy control is changing this situation, and opening up the use of vague data. This makes it possible in many cases to build controls that are more robust, cheaper and require less energy to operate. Indeed, fuzzy logic is the only possible answer to a number of challenging control problems.

Since 1987 the Japanese city of Sendai has had a driverless subway system that is automatically operated by a so-called fuzzy controller. Whereas in Europe the term "fuzzy" has long aroused negative associations, the Japanese are increasingly embracing the concept and applying it. For instance, fuzzy logic can decide on the optimum time for a car to shift gears, can manage the amount of suction needed by a vacuum cleaner, and can even limit subject movement in video cameras. But now the fuzzy wave has also reached Europe.

Fuzzy Control

What is a fuzzy controller? As far as the operation of a subway system is concerned, the problem can be simplified as follows: the positions of accelerator lever and brake lever must be determined on the basis of available measured data (e.g. current speed, position) and desired targets (e.g. required speed curve). Basically, there are three possible ways of achieving this (Fig. 1). The most widespread method is that of manual operation, i.e. the translation of measured data and target requirements into acceleration and braking actions by the driver. If, however, we wish to automate this operation (one possible way of increasing the frequency of trains in local public transport), the classical approach is based on the following principle: mathematical models are used to provide as accurate a description as possible of the technical process controlled by the driver, and this model is then used as the basis for algorithmic methods, such as so-called PID controllers. Conversely, with fuzzy control, it is not the technical system that is modeled, but the manner in which a human process controller acts, i.e. how the driver drives the train.

But how is a subway train driven? Interviews with drivers and technicians result in the formulation of rules such as the following: If the train is a short distance from the station and is traveling at average speed, then an average braking force is required.

A central problem in this respect is the term "average speed," which must be described in formal terms so that such a rule can be processed in a computer. The first possible solution to this problem is shown in Figure 2: the normal speed range of a subway train is broken down into sections in each of which a clear definition is made: yes, 40 km/h is an average speed or no, 39 km/h is not an average speed. Such a solution entails two problems. No subway driver or technician is able with certainty to draw a precise dividing line between what is and what is not an average speed. Even if such unambiguously defined limits were available, controls based on these would result in a jerky ride at the points of transition between one speed range and the next, since the above rule, for example, is not applied at all at 39 km/h but is wholly enforced at 40 km/h.

This is where fuzzy logic and fuzzy control enter the picture. Such systems make it possible to produce a gradual transition in speed, as shown in Figure 2. There are speed ranges in which the question "Is this an average speed?" can be clearly answered with yes or no; the transitions between ranges, however, are fluid or fuzzy. A speed of 40 km/h corresponds only to a certain extent to a subway driver's concept of an average speed – and only to that extent will a rule responding to such a speed be satisfied and applied. It is important to note that fuzzy control does not necessarily have anything to do with fuzzy data, but with fuzzy control concepts used in the processing of data – of both the fuzzy and non-fuzzy kind.

Typically, a fuzzy control consists of 20 to 100 such rules that are run through in a loop. Measured data and reference variables are inputted into the control at defined intervals; the output from the control comprises control actions or manipulated variables derived using these rules. Consequently, a fuzzy controller is a real-time expert system used in process automation that employs fuzzy logic in order to represent qualitative variables. Both the gradual decision-making functions and the rules and their execution are coupled to very elementary operations, which provides the basis for specific software and hardware support (fuzzy chips), and thus permits efficient, real-time-capable solutions. Considering fuzzy controllers as real-time expert systems, their relation to neuronal networks is of particular interest. Both systems are based on the same principle: they attempt to model human thought processes and, in particular, the soft decisions that occur in such

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The San Diego Light Rail System: Facts & Figures

Overview: Two lines, 32 miles of track, 71 Siemens Duewag U2 light rail vehicles (with 75 more on order) powered by 600-V DC overhead lines, 33 stations, over 4000 free parking places at 16 park-and-ride lots. Total personnel: 264.

Operating performance: 50,000 passengers per weekday, over 15 million passengers carried in 1990, 93.15 percent farebox recovery rate, 98.9 percent on time.

Fares: Self-service system, random inspection by roving fare inspectors, one percent evasion rate. Fares range from 50 cents to $2.00 one way, depending on distance; multi-ride, day-tripper, and monthly passes available.

Organization: The city's light rail system is run by San Diego Trolley, Inc (SDTI), which is a wholly owned subsidiary of the Metropolitan Transit Development Board (MTDB). The Board's area of jurisdiction covers approximately 570 square miles, and a population of 1.7 million.

***SR:** What was the process that led up to the decision to invest in a light rail system here in San Diego?*

Senator Mills: It was a very simple process. As the president of the Senate of California, I carried Senate Bill 101, which created the San Diego Metropolitan Transit Development Board – MTDB –, gave it the responsibility of building a rail system, and the money to implement that decision. There wasn't any general public, government, or business support for it at the time. But when the money was provided, it was spent, and that created the south leg of our transit system. Today, there's general support for the system.

***SR:** Once the decision had been approved, how did the San Diego MTDB go about awarding contracts for the rolling stock, and why did it decide on Duewag?*

Senator Mills: One of the provisions of Senate Bill 101 was that only service-proven equipment could be purchased for the city's public transportation. That provision was designed to avoid the kind of costly experimentation that has taken place in some other American cities. With that in mind, the decision to buy Duewag equipment in particular stemmed from a field trip by MTDB members to Edmonton and Frankfurt. In those cities the Board members saw U2 cars in operation. They were favorably impressed by the appearance, characteristics, and maintenance records of the vehicles, as well as the statements of the operators. The result was that a decision was made to buy U2 cars.

***SR:** The San Diego Light Rail System is considered to be the most successful in the United States. What makes it superior to other systems?*

Senator Mills: It is the most successful of the new light rail systems in the United States for two reasons. Number one, from a financial point of view, it has performed better than any other system. It has covered a larger percentage of operating costs out of the farebox than any other passenger railway in the United States. San Diego's system also costs less per passenger mile to run than any other urban passenger railway in the United States.

The second point is that the system has generated more patronage than any other new system by a wide margin. We are now carrying 50,000 people per day on weekdays, and the figure is steadily increasing. Since January 1983 we have never had a month of operations in which we didn't carry at least as many people as in the same month the year before. On the south line we are now carrying about three times as many people as when the line opened.

***SR:** What accounts for this? Has the MTDB promted the system in the media?*

Senator Mills: There has been virtually no promotional effort. We once bought a billboard. We've only paid for advertising twice. That's it. Other than that we do a good job of public relations and public education. But for the most part ridership grows as a result of favorable reports made by patrons.

***SR:** You said the system costs less to operate than any other system in the United States. What accounts for that?*

Senator Mills: I should have added that the system cost less to build than any other. You see, there was a concerted effort to keep costs down. The stations were built inexpensively. The repair facilities are inexpensive, prefabricated buildings. The signalling system is traditional. It's a simpler system than many others. This keeps capital costs and operating costs relatively low; the latter is particularly important because personnel account for 50 percent of our expenses.

***SR:** Major sections of the line network were created by rebuilding freight train track, and today the city's rail system is being used by both light rail vehicles and freight trains. What economic advantages has the city derived from this dual use strategy?*

Senator Mills: This was an important innovation. It was a very low-cost way of acquiring right-of-way for the passenger railway, while at the same time maintaining rail connections for many local industries. Most American cities have rail lines running through them, but surprisingly, none have taken advantage of them as San Diego has.

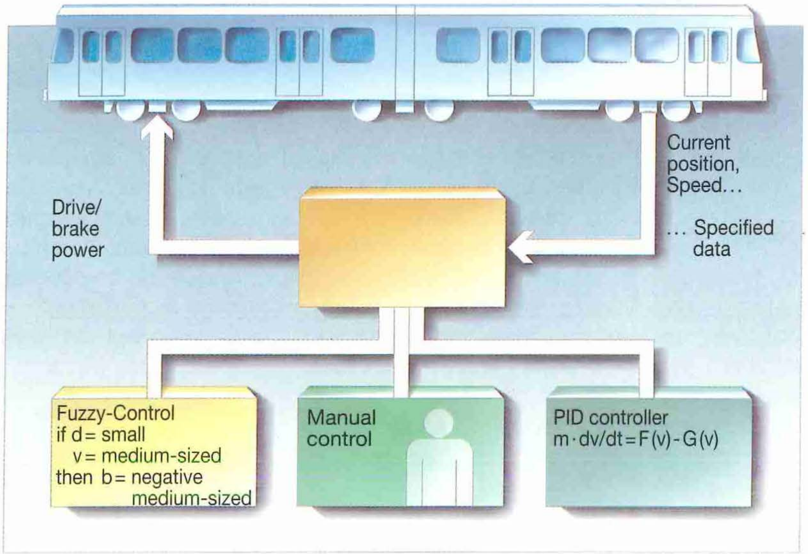
***SR:** Your rolling stock is remarkably clean and free of graffiti. What measures has the MTDB taken to ensure such a high level of maintenance?*

Senator Mills: Our policy is to take any graffiti off the cars as soon as it's discovered. When a car completes a run it's checked. If graffiti is found, the car is taken out of service at that point; not an hour later, and not at the end of the day. The result is that people don't put graffiti on. People who do that kind of thing like to see their work. But if they never see it again they don't have the motivation to do it in the first place. The same maintenance strategy is applied to our stations. And the reasoning behind the strategy is simple: graffiti has an adverse effect on ridership.

***SR:** Looking ahead, how does San Diego intend to further expand and improve its light rail network, and is there a general policy or philosophy with respect to limiting the use of private vehicles?*

Senator Mills: The money to double the size of our light rail system is already available. We recently ordered 75 light rail vehicles from Duewag corporation with a value of $120 million to serve the first of our planned extensions. We will be expanding the system to the north, and building another line to the east. We expect that the system will grow at a faster rate over the next ten years than it has over the last ten. And I imagine we will find funding to continue expanding the system thereafter.

At present there is no general policy or philosophy with respect to limiting the use of private vehicles. That is in the works, however, – the air pollution control board, which was created by state law, is considering that and is expected to produce policies that will limit the use of private vehicles. In particular it will place penalties on employers who fail to reduce the number of employees who come to work in private vehicles. ●



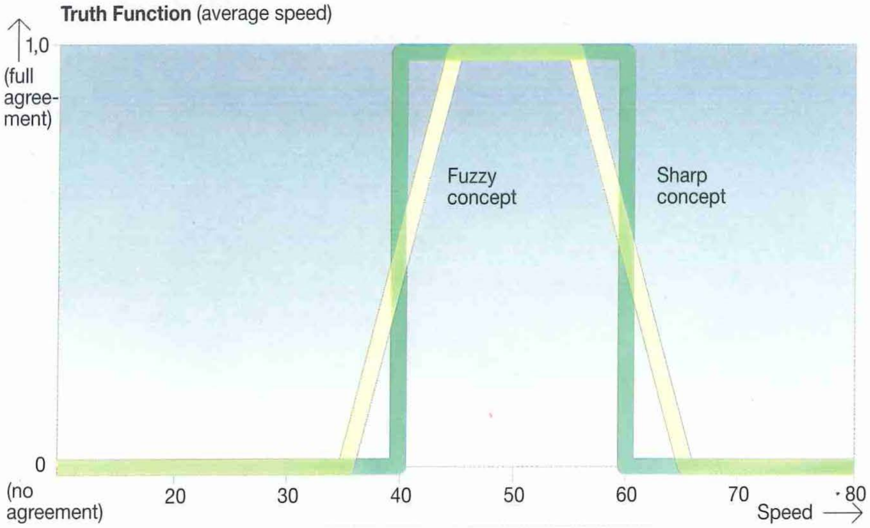


Fig. 1
There are three ways of controlling a train: manually, with conventional automation, and with fuzzy control

Fig. 2
Average speed as a clearly defined concept and as represented in fuzzy control (yellow, with fluid transitions)

processes. In the case of neuronal networks, this is done at the level of so-called damp hardware, i.e. neuronal computing structures are imitated. Fuzzy control operates at a somewhat more abstract cognitive level.

A Vast Range of Applications

Assessments of the potential of fuzzy control vary between euphoria and extreme reservation. ler assessment is in keeping with the state of the art. To be sure, many automation problems can be satisfactorily solved using conventional techniques and, apart from prestige or marketing slogans, provide no technological arguments in favor of the use of fuzzy control. There are, however, a wide range of problems where, although a solution can be achieved by conventional methods, fuzzy logic could provide real advantages. This occurs typically in cases where there is no accurate or simple mathematical model of the system in question, because it is precisely such a model that forms the basis for conventional solutions. Fuzzy control, however, does not need such a model, but simulates the strategy of the person controlling a process. Thus, in such cases, this is the only method that makes it possible to arrive at satisfactory solutions. Typical applications are industrial processes such as cement production, sewage treatment, or general environmental engineering, where efficient models are frequently not available. In the medium and long term, these will definitely become important fields of application for fuzzy control.

On a case-by-case basis, however, fuzzy control can also be seen as being in direct competition with conventional solutions, as in the previously described control system for subway trains. This is an area where fuzzy control frequently provides advantages as regards the quality of the solution or as regards development or implementation costs. For instance, fuzzy control enables:

- trains to travel in a smooth and energy-saving manner;
- washing machines to make do with 4-bit processors, whereas conventional controls require 8-bit processors;
- air conditioning systems to be built with very great flexibility in comparatively short periods, whereas conventional controls have to be extensively adapted to varying conditions.

A review conducted in the Spring of 1989 revealed over 120 successful industrial applications of fuzzy control, with the overwhelming majority being in Japan. To date, accurate, comprehensive studies of fuzzy control's market potential have either not been made or are not publicly available. Two figures, however, provide an indication of how fuzzy control is rated by Japanese companies: for the mid-'90s Omron expects to achieve over $1 billion worth of sales annually with fuzzy-control products and, in the next few years, Panasonic is aiming for the "fuzzification" of some 200 products.

Needed: Parameters for Fuzzy Applications

In spite of the apparent success of fuzzy control systems, the technology has shortcomings that still represent an obstacle to its full exploitation. For example, although there is a great deal of experience as regards those applications in which fuzzy control can be put to good use, there is still no established system that makes it possible, on the basis of problem-specific characteristics, to decide whether, in which version, and with what benefits fuzzy control should be employed.

The development of an all-embracing theory of fuzzy control and, in particular, of a resultant, systematic method of development is still a long way from completion. This is also expressed in the fact that software-development environments for the implementation of fuzzy-control applications scarcely exceed simple, graphics-oriented rule editors and compilers. While there are specific software- or hardware-supported routine environments for fuzzy-control applications providing satisfactory performance and an improving price/performance ratio, the economical application of fuzzy control in a broader range of applications depends on the further development of systematic design techniques and efficient development tools.

In addition, a shortage of personnel trained in this technology is one of the principal obstacles to the broad-based use of fuzzy control in Germany. So far, the subject has hardly been touched in German universities. There are only very few experts, and the available literature provides engineers and information scientists with very little assistance as far as the design and implementation of fuzzy-control applications is concerned. Thus, overall, there are three areas in which progress is still required:

- systematic design and implementation methods,
- powerful software tools, and
- improved training.

Worldwide R&D Activities

Around the world, but particularly in Japan, intense work is being conducted on overcoming the above-outlined shortcomings. One of the centers of Japanese activity is the LIFE Institute (Laboratory for International Fuzzy Engineering Research) in Yokohama. Modeled after the Fifth Generation Institute, the LIFE Institute not only enjoys an annual budget of $10 million but also has many industrial member companies, including Canon, Fuji Heavy, Hitachi, Matsushita Electric, Mitsubishi Electric, IBM Japan, and Thomson Japan. In addition, several universities as well as companies such as Omron (according to its own information, this company has some 50 employees working on fuzzy control) are pursuing their own, extensive activities in this area. While Japanese companies have so far largely restricted themselves to their domestic market, they are now attempting to gain a foothold in the European market with fuzzy-control products.

In addition to Japan, India and China are very actively engaged in fuzzy research, as is demonstrated by their representation in the International Fuzzy Set Association. While there are fewer than 500 members each for Europe and the U.S., China and India each have 2000 members.

In the U.S., the aerospace industry – chiefly Boeing and NASA – has shown the greatest interest in the field of fuzzy control. Currently, the most important manufacturer of fuzzy-control hardware and software outside Japan is Togai Infralogic, Inc., a software company whose main products are a development system for efficient fuzzy-control applications (based on standard microprocessors), as well as special fuzzy accelerator boards.

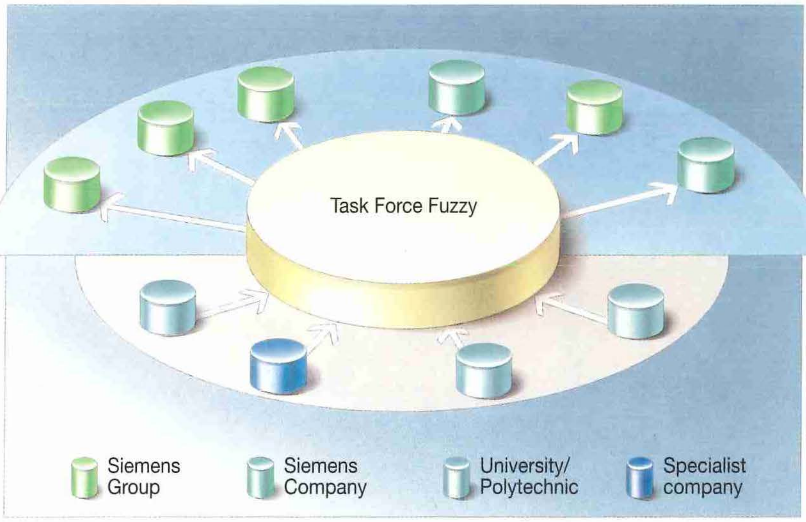
In Germany, as in the rest of Europe, fuzzy control has so far been pursued predominantly as a theoretical discipline. To date, however, it has rarely been translated into applications. A notable exception to this is described in the adjoining article. However, many companies are now beginning to examine fuzzy control – in most cases with small-scale evaluation projects.

Fuzzy Control at Siemens

Fuzzy control can be used advantageously in many areas of automation in which Siemens is involved. These include knowledge-based systems, control engineering, electronic circuits, and corresponding fields of application.

The foundations for work in these and other areas were laid by Task Force Fuzzy, which was established in January 199 Siemens' Central Research and Development Department (Fig. 3). Task Force Fuzzy's ten members are pursuing two essential objectives:

Fig. 3
Task Force Fuzzy works closely with internal and external partners



first, in collaboration with the company's product divisions, the Task Force is pursuing pilot applications capable of rapid development. Secondly, in-house improvements to methods and tools for fuzzy control are being examined. These are aimed at the systematic support of the entire life cycle of a fuzzy-control application from initial design through acquisition of the knowledge base to startup and optimization.

Task Force Fuzzy is working with application experts inside the company, as well as with numerous external partners. Projects have been designed in such a way that pilot applications are implemented by teams, with the Task Force providing the expertise in fuzzy control and the respective product di[illegible]n supplying the specific application expertise.

Other internal partners include hardware and sensor technology experts from the central microelectronics-development department and from the semiconductors product division. The resulting synergies are expected to lead to decisive competitive advantages.

Outside the company, Task Force Fuzzy is working with manufacturers of fuzzy-control hardware and software. For example, joint application projects are underway with, among others, Togai Infralogic. On the research side, there is close cooperation with the Institute for Corporate Research at RWTH (a German technical university) as well as with other German and international research institutes. Thus, in overall terms, Task Force Fuzzy has been established as a Corporate Center of Competence. The Task Force operates as the interface between [illegible]nal partners and Siemens' product divisions and thus, in addition to its own personnel, supports more extensive R&D activities relating to fuzzy control. ●

Developing Fuzzy Control Technology in Europe

Hans-Jürgen Zimmermann

In 1965 the first publication on fuzzy sets appeared in the U.S., and in the mid-'70s the first fuzzy controller was introduced in Europe. Yet even by the end of the '70s, Japan displayed no apparent interest in fuzzy control technology. Japan's massive advance in a broad range of fuzzy products thus requires a 4-point explanation:

- The seeds of this technology were randomly dispersed throughout the world. In the U.S. they took root in information science, an attractive, but not very application-friendly area. Similarly, in Europe they took root above all in the field of operations research. But in Japan, control engineers seized on the new approach and quickly used it to solve concrete problems.
- The close relationships between universities and industry in Japan promote the rapid conversion of ideas into products: university laboratories are financed to a large extent by industry. Thus knowledge gained in the course of R&D activities is quickly brought to market.
- The strategic thinking of the Japanese, which is geared to long-term goals, creates a solid basis for the rapid implementation of innovative ideas.
- The technology-friendly mentality of Japanese consumers has given a boost to fuzzy fever in Japan.

The fact that similar interest has developed in Germany as well, at least among potential manufacturers, is due in large measure to the media.

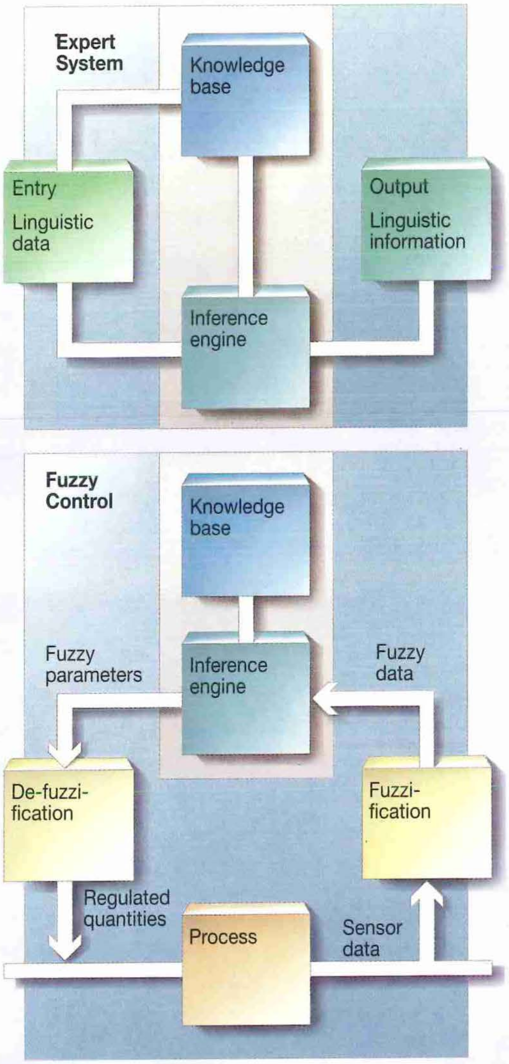


Fig. 1
The basic structure of expert and fuzzy control systems is very similar: the knowledge base and inference engine form the core; in both cases non-numerical data must be processed

The Potential for Fuzzy Products

The U.S. and Europe are clearly out in front as regards scientific know-how in this field. This is not surprising in view of the heavy engineering bias of the Japanese.

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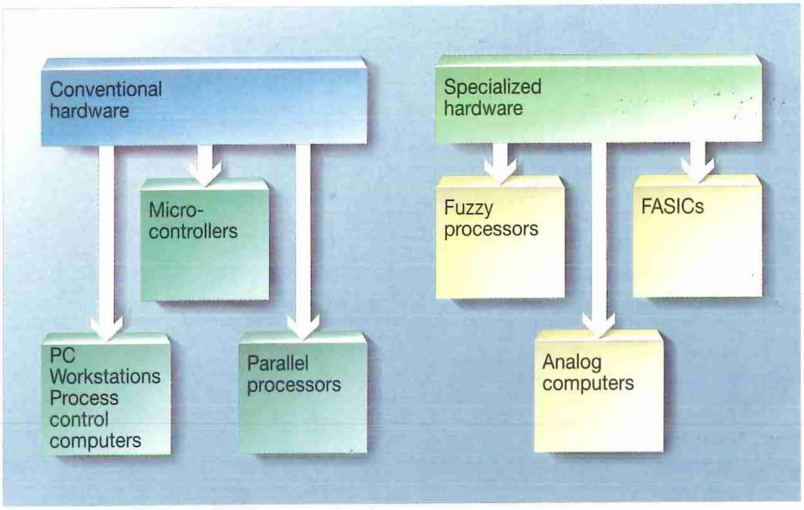
Possible Hardware Platforms for Fuzzy Control



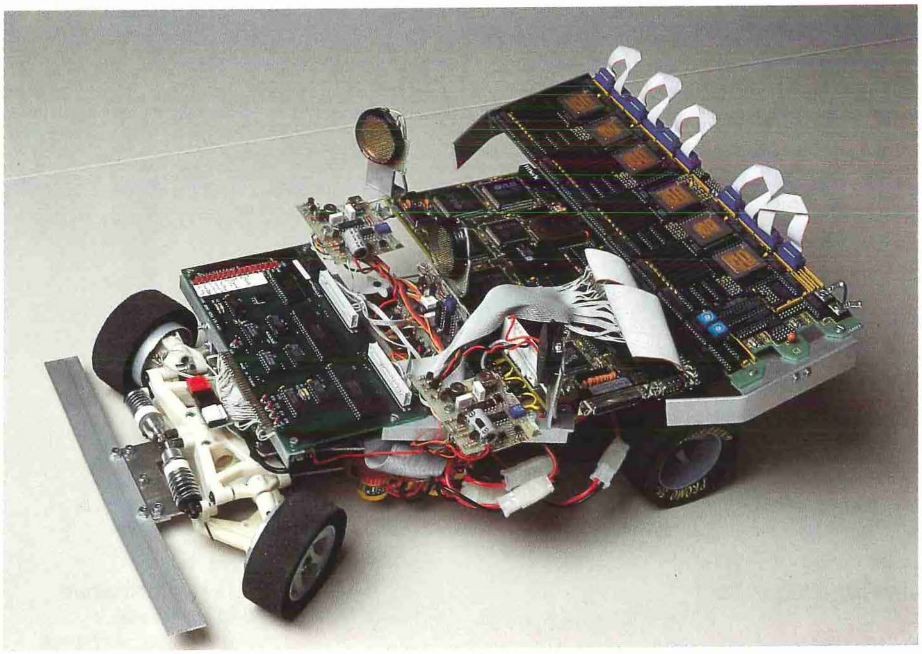


Fig. 2
Fuzzy applications extend from PCs via parallel processors, microcontrollers, and special fuzzy processors to application-specific fuzzy chips (FASICs)

Fig. 3
Entry of the knowledge base for this fuzzy car took only two hours

Conversely, practical experience – especially in fuzzy control – is still very thin in Europe, where there is a distinct shortage of experts.

The potential fields of application for fuzzy control lie predominantly in highly complex tasks involving large volumes of data. Hardware, software, and brainware, as well as scientific and practical know-how contribute significantly to market opportunities for fuzzy products. The range of available hardware extends from the conventional digital computer through parallel computers, microcontrollers and analog or digital fuzzy processors to application-specific fuzzy chips (FASICs). There is already a wide array of software tools for implementing fuzzy control systems; many, however, have not progressed beyond the pilot-version stage. Brainware – the generic term for methods, theories, and techniques – has so far been based on a relatively simple foundation, some of which was laid as early as the '60s.

Can Europe Challenge Japan?

Europe's chances of challenging Japan can be characterized by the following eight points:

- Japan has a three- to four-year lead as regards practical applications.
- Japanese experience is concentrated on control-related applications; as regards fuzzy data ana or the use of expert syste Europe is in front.
- The scientific base is broader and deeper in Europe.
- Europe and the USA are more heavily software-oriented.
- Japan is entering the world market with fuzzified products and fuzzy hardware; the USA is concentrating more on software.
- Europe should promote fuzzy products and solutions instead of fuzzy techniques.
- The existence of a valuable pool of German ideas is confirmed by the large number of inquiries to RWTH Aachen concerning the possibility of making fuzzy products.
- The German fuzzy-control markets and the world markets are currently being divided up; the Japanese market is already firmly in Japanese hands. Strong American competition can be expected as of 1992.

To sum up: At present, E peans still have a perfectly good opportunity to challenge Japan. But there is very little time left to seize that opportunity. ●